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An Inequality of Simpson's type via Quasi–Convex Mappings with Applications

Research Article

Mohammad W. Alomari*

Department of Mathematics, Faculty of Science and Information Technology, Irbid National University, 21110 Irbid, Jordan.

Abstract: In this paper, an inequality of Simpson type for quasi-convex mappings is proved. The constant in the classical Simpson's inequality is improved. Furthermore, the obtained bounds can be (much) better than some recent obtained bounds. Application to Simpson's quadrature rule is also given.

MSC: Primary 26D15; Secondary 65D30; 65D32

Keywords: Simpson's inequality • quadrature • quasi-convex

1. Introduction

Suppose $f : [a,b] \to \mathbb{R}$, is fourth times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty := \sup\limits_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. The following inequality

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty \tag{1}$$

holds, and it is well known in the literature as Simpson's inequality. It is well known that if the mapping f is neither four times differentiable nor is the fourth derivative $f^{(4)}$ bounded on (a,b), then we cannot apply the classical Simpson quadrature formula.

In [10], Pečarić and Varošanec, obtained some inequalities of Simpson's type for functions whose n-th derivative, $n \in \{0,1,2,3\}$ is of bounded variation, as follow:

* E-mail: mwomath@gmail.com

Theorem 1.1.
Let $n \in \{0,1,2,3\}$. Let f be a real function on $[a,b]$ such that $f^{(n)}$ is function of bounded variation. Then

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \le C_n (b-a)^{n+1} \bigvee_a^b \left(f^{(n)} \right), \tag{2}$$

where,

$$C_0 = \frac{1}{3},\ C_1 = \frac{1}{24},\ C_2 = \frac{1}{324},\ C_3 = \frac{1}{1152},$$

and $\bigvee_a^b \left(f^{(n)} \right)$ is the total variation of $f^{(n)}$ on the interval $[a,b]$.

Here to note that, the inequality (2) with $n = 0$, was proved in literature by Dragomir [5]. Also, Ghizzetti and Ossicini [9], proved that if f''' is an absolutely continuous mapping with total variation $\bigvee_a^b (f)$, then (2) with $n = 3$ holds.

In recent years many authors were established several generalizations of Simpson inequality for mappings of bounded variation, Lipschitzian, monotonic, and absolutely continuous mappings via kernels, for refinements, counterparts, generalizations and several Simpson's type inequalities see [4]–[16].

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$, is said quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda) y) \le \sup \{ f(x), f(y) \},$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex nor continuous.

Example 1.1.
The floor function $f_{loor}(x) = \lfloor x \rfloor$, is the largest integer not greater than x, is an example of a monotonic increasing function which is quasi-convex but it is neither convex nor continuous.

In the same time, one can note that the quasi-convex mappings may be not of bounded variation, i.e., there exist quasi-convex functions which are not of bounded variation. For example, consider the function $f : [0,2] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x \sin\left(\frac{\pi}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases},$$

therefore, f is quasi-convex but not of bounded variation on [0,2]. Therefore, we cannot apply the above inequalities. For recent inequalities concerning quasi-convex mappings see [1]–[4].

In this paper, we obtain an inequality of Simpson type via quasi-convex mapping. This approach allows us to investigate Simpson's quadrature rule that have restrictions on the behavior of the integrand and thus to deal with larger classes of functions. In general, we show that our result is better than the classical inequality (1).

2. Inequalities of Simpson's type for quasi-convex functions

In order to prove our main results, we start with the following lemma:

Lemma 2.1.
Let $f''' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, where $a, b \in I$ with $a < b$. If $f^{(4)} \in L[a,b]$, then the following equality holds:

$$\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = (b-a)^5 \int_0^1 p(t)\, f^{(4)}(ta + (1-t)b)\,dt, \tag{3}$$

where,

$$p(t) = \begin{cases} \frac{1}{24}t^3\left(t - \frac{2}{3}\right), & t \in \left[0, \frac{1}{2}\right] \\ \frac{1}{24}(t-1)^3\left(t - \frac{1}{3}\right), & t \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Proof. We note that

$$\begin{aligned} I = (b-a)^5 \int_0^1 p(t)\, f^{(4)}(ta + (1-t)b)\,dt &= \frac{(b-a)^5}{24}\int_0^{1/2} t^3\left(t - \tfrac{2}{3}\right) f^{(4)}(ta + (1-t)b)\,dt \\ &+ \frac{(b-a)^5}{24}\int_0^{1/2} (t-1)^3\left(t - \tfrac{1}{3}\right) f^{(4)}(ta + (1-t)b)\,dt. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I = {} & \tfrac{1}{24}t^3\left(t - \tfrac{2}{3}\right)\frac{f^{(3)}(ta + (1-t)b)}{a-b}\bigg|_0^{1/2} - \left(\tfrac{1}{6}t^3 - \tfrac{1}{12}t^2\right)\frac{f''(ta + (1-t)b)}{(a-b)^2}\bigg|_0^{1/2} \\ & + \left(\tfrac{1}{2}t^2 - \tfrac{1}{6}t\right)\frac{f'(ta + (1-t)b)}{(a-b)^3}\bigg|_0^{1/2} - \left(t - \tfrac{1}{6}\right)\frac{f(ta + (1-t)b)}{(a-b)^4}\bigg|_0^{1/2} \\ & + \int_0^{1/2}\frac{f(ta + (1-t)b)}{(a-b)^4}dt + \tfrac{1}{24}(t-1)^3\left(t - \tfrac{1}{3}\right)\frac{f^{(3)}(ta + (1-t)b)}{a-b}\bigg|_{1/2}^{1} \\ & - \left(\tfrac{1}{6}t^3 - \tfrac{5}{12}t^2 + \tfrac{1}{3}t - \tfrac{1}{12}\right)\frac{f''(ta + (1-t)b)}{(a-b)^2}\bigg|_{1/2}^{1} \\ & + \left(\tfrac{1}{2}t^2 - \tfrac{5}{6}t + \tfrac{1}{3}\right)\frac{f'(ta + (1-t)b)}{(a-b)^3}\bigg|_{1/2}^{1} - \left(t - \tfrac{5}{6}\right)\frac{f(ta + (1-t)b)}{(a-b)^4}\bigg|_{1/2}^{1} \\ & + \int_{1/2}^{1}\frac{f(ta + (1-t)b)}{(a-b)^4}dt \end{aligned}$$

Setting $x = ta + (1-t)b$, and $dx = (a-b)dt$, gives

$$(b-a)^5 \cdot I = \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

which gives the desired representation (3). $\square$

Therefore, we may state our main result as follows:

Theorem 2.1.
Let $f''': I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $f^{(4)} \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f^{(4)}\right|$ is quasi-convex on $[a,b]$, then the following inequality holds:

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ \leq \frac{(b-a)^5}{5760}\left[\sup\left\{\left|f^{(4)}(a)\right|, \left|f^{(4)}\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|, \left|f^{(4)}(b)\right|\right\}\right]. \quad (4)$$

Proof. From Lemma 2.1, and since f is quasi-convex, we have

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\
&= \left|(b-a)^5 \int_0^1 p(t)\, f^{(4)}(ta + (1-t)b)\,dt\right| \\
&\leq (b-a)^5 \int_0^1 |p(t)| \left|f^{(4)}(ta + (1-t)b)\right| dt \\
&= (b-a)^5 \int_0^{1/2} |p(t)| \left|f^{(4)}(ta + (1-t)b)\right| dt \\
&\quad + (b-a)^5 \int_{1/2}^1 |p(t)| \left|f^{(4)}(ta + (1-t)b)\right| dt \\
&\leq (b-a)^5 \int_0^{1/2} \left|\tfrac{1}{24}t^3\left(t - \tfrac{2}{3}\right)\right| \cdot \sup\left\{\left|f^{(4)}(b)\right|, \left|f^{(4)}\left(\frac{a+b}{2}\right)\right|\right\} dt \\
&\quad + (b-a)^5 \int_{1/2}^1 \left|\tfrac{1}{24}(t-1)^3\left(t - \tfrac{1}{3}\right)\right| \cdot \sup\left\{\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|, \left|f^{(4)}(a)\right|\right\} dt \\
&= \frac{(b-a)^5}{5760}\left[\sup\left\{\left|f^{(4)}(a)\right|, \left|f^{(4)}\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|, \left|f^{(4)}(b)\right|\right\}\right],
\end{aligned}$$

which completes the proof. □

Corollary 2.1.
Let f as in Theorem 2.1.

1. *If f is decreasing, then we have*

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(b-a)^5}{5760}\left[\left|f^{(4)}(a)\right| + \left|f^{(4)}\left(\frac{a+b}{2}\right)\right|\right]. \quad (5)$$

2. *If f is increasing, then we have*

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(b-a)^5}{5760}\left[\left|f^{(4)}\left(\frac{a+b}{2}\right)\right| + \left|f^{(4)}(b)\right|\right]. \quad (6)$$

Corollary 2.2.
Let f as in Theorem 2.1. If $f^{(4)}$ is exits, continuous and $\left\|f^{(4)}\right\|_\infty := \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, then the inequality (4) reduced to (1).

Remark 2.1.
1- We note that the constants in (1) and (2) are improved.

2- The corresponding version of the inequality (4) for powers may be done by applying the Hölder inequality and the power mean inequality.

Example 2.1.
Let $f(x) = \sin x$, $x \in [0, \frac{\pi}{2}]$, then $f(0) = 0$, $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, $f(\frac{\pi}{2}) = 1$. Now, by applying the classical Simpson's inequality (1), we get

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = \frac{\pi}{12}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + f(1)\right] + \frac{\pi^5}{32} \cdot \frac{1}{2880} = 1.005600403.$$

However, by applying (4) or (6) since f is increasing on $[0, \frac{\pi}{2}]$, we get

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = \frac{\pi}{12}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + f(1)\right] + \frac{\pi^5}{32} \cdot \frac{1}{2880} = 1.005114123.$$

Comparing with the exact value which is

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = 1$$

we deduce that our result (6) is better than (1) for this example. So that, in general (4) can be better than (1).

3. Applications to Simpson's Formula

Let d be a division of the interval $[a, b]$, i.e., $d : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, $h_i = (x_{i+1} - x_i)/2$ and consider the Simpson's formula

$$S(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i + h_i) + f(x_{i+1})}{6}(x_{i+1} - x_i). \tag{7}$$

It is well known that if the mapping $f : [a, b] \to \mathbf{R}$, is differentiable such that $f^{(4)}(x)$ exists on (a, b) and $M = \sup_{x \in (a,b)} \left|f^{(4)}(x)\right| < \infty$, then

$$I = \int_a^b f(x)\,dx = S(f, d) + E_S(f, d), \tag{8}$$

where the approximation error $E_S(f, d)$ of the integral I by the Simpson's formula $S(f, d)$ satisfies

$$|E_S(f, d)| \le \frac{M}{2880} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^5. \tag{9}$$

In the following we give a new estimation for the remainder term $E_S(f, d)$.

Proposition 3.1.
Let $f''': I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $f^{(4)} \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f^{(4)}\right|$ is quasi-convex on $[a,b]$, then in (8), for every division d of $[a,b]$, the following holds:

$$|E_S(f,d)| \leq \frac{1}{5760}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^5\left[\sup\left\{f^{(4)}(x_i), f^{(4)}\left(\frac{x_i+x_{i+1}}{2}\right)\right\} + \sup\left\{f^{(4)}\left(\frac{x_i+x_{i+1}}{2}\right), f^{(4)}(x_{i+1})\right\}\right].$$

Proof. Applying Theorem 2.1 on the subintervals $[x_i, x_{i+1}]$, $(i = 0,1,...,n-1)$ of the division d, we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{(x_{i+1}-x_i)}{6}\left[f(x_i)+4f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]\right| \leq (x_{i+1}-x_i)^5\left[\sup\left\{\left|f^{(4)}(x_i)\right|, \left|f^{(4)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\} + \sup\left\{\left|f^{(4)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f^{(4)}(x_{i+1})\right|\right\}\right]$$

Summing over i from 0 to $n-1$ and taking into account that $\left|f^{(4)}\right|$ is quasi-convex, we deduce that

$$\left|\int_a^b f(x)\,dx - S(f,d)\right| \leq \frac{1}{5760}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^5\left[\sup\left\{\left|f^{(4)}(x_i)\right|, \left|f^{(4)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\} + \sup\left\{\left|f^{(4)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f^{(4)}(x_{i+1})\right|\right\}\right].$$

which completes the proof. □

4. conclusion

For fourth times continuously differentiable mapping $f$on (a,b) and $\left\|f^{(4)}\right\|_{\infty} := \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, the classical Simpson's inequality holds. In this paper, we relax the conditions on Simpson's inequality; namely, the proposed inequality (4) holds if $f''': I \subseteq \mathbb{R} \to \mathbb{R}$ is an absolutely continuous mapping on I° such that $f''' \in L[a,b]$ and $\left|f^{(4)}\right|$ is quasi-convex on $[a,b]$. In general, the bound in (4) is better than that one in the classical Simpson's inequality (1), as shown in Example 1.

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