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On Gautschi's conjecture on subrange Jacobi polynomials

Research Article

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Abstract: In this short note, we give a partial positive answer to Gautschi's conjecture about the monotonicity of positive zeros of subrange Jacobi polynomials, stated recently in his paper [Numer. Algorithms **79** (2018), no. 3, 759–768].

MSC: 33C47, 65D30

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1. Introduction

Recently Walter Gautschi [6] has considered zeros of (monic) subrange Jacobi polynomials of degree n, in notation $\pi_n(\cdot) = \pi_n^{(\alpha,\beta)}(\cdot\,;c)$, which are orthogonal on $[-c,c]$, $0<c<1$, with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta>-1$. Such kind of orthogonal polynomials on a strict subinterval of $[-1,1]$, including their numerical computation, as well as the related Gaussian quadrature rules, were introduced and studied also by Gautschi in his earlier paper [5]. It is interesting that in [2] and [3] Da Fies and Vianello, in connection with subperiodic trigonometric quadrature, used sub-range Chebyshev polynomials orthogonal with respect to the Chebyshev weight of the first kind ($\alpha=\beta=-1/2$) on the interval $[-c,c]$, where $c=\sin(\omega/2)$, with $0<\omega<\pi$, in order to construct some kind of Gaussian product formulas for integration over circular and annular sectors, circular zones, etc.

As in [7], in his study of the monotonicity behavior of the zeros x_ν of the orthogonal polynomial $\pi_n^{(\alpha,\beta)}(x;c)$, Gautschi started from the respective Gaussian quadrature formula,

$$\int_{-c}^{c} p(x)w(x)\mathrm{d}x = \sum_{\nu=1}^{n} A_\nu(c)p(x_\nu(c)), \quad p\in\mathcal{P}_{2n-1}, \tag{1}$$

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where $x_\nu = x_\nu(c)$ are the zeros of the nth-degree subrange Jacobi polynomial $\pi_n^{(\alpha,\beta)}(x;c)$ and $A_\nu(c)$ are the corresponding weight coefficients (Christoffel numbers). Then, differentiating (1) with respect to c,

$$p(c)w(c) + p(-c)w(-c) = \sum_{\nu=1}^{n} \frac{\mathrm{d}A_\nu(c)}{\mathrm{d}c}\, p(x_\nu(c)) + \sum_{\nu=1}^{n} A_\nu(c)p'(x_\nu(c))\,\frac{\mathrm{d}x_\nu}{\mathrm{d}c},$$

using A. Markov idea (cf. [8, §6.12]), and putting $p(x) = [\pi_n(x)]^2/(x - x_\nu)$ ($p \in \mathcal{P}_{2n-1}$, because $\pi_n(x_\nu) = 0$), he concluded that

$$w(c)\pi_n(c)^2 \left\{ \frac{1}{c - x_\nu} - \left[\frac{\pi_n(-c)}{\pi_n(c)}\right]^2 \frac{w(-c)}{w(c)} \cdot \frac{1}{c + x_\nu} \right\} = A_\nu(c)[\pi_n'(x_\nu)]^2 \frac{\mathrm{d}x_\nu}{\mathrm{d}c}, \tag{2}$$

because for all ν, $p(x_\nu) = 0$ and $p'(x_\nu) = [\pi_n'(x_\nu)]^2$.

In the symmetric ultraspherical case ($\alpha = \beta$), when $w(-c) = w(c)$ and $\pi_n(-c)^2 = \pi_n(c)^2$, (2) reduces to

$$w(c)\pi_n(c)^2 \frac{2x_\nu}{c^2 - x_\nu^2} = A_\nu(c)[\pi_n'(x_\nu)]^2 \frac{\mathrm{d}x_\nu}{\mathrm{d}c},$$

wherefrom for $x_\nu > 0$, Gautschi concluded that $\mathrm{d}x_\nu/\mathrm{d}c > 0$ and, in this way, he proved that *all positive zeros x_ν of the subrange ultraspherical polynomials ($\alpha = \beta$), orthogonal on $[-c, c]$, $0 < c < 1$, are monotonically increasing as functions of c* (see [6, Theorem 1]).

In the general case when $-1 < \alpha < \beta$, for any $n \in \mathbb{N}$ and any c with $0 < c < 1$, because of $\pi_n^{(\alpha,\beta)}(x;c) = \pi_n^{(\beta,\alpha)}(-x;c)$ and

$$\frac{w(-c)}{w(c)} = \left(\frac{1-c}{1+c}\right)^{\beta-\alpha},$$

(2) reduces to

$$w(c)\pi_n(c)^2 \left\{ \frac{1}{c - x_\nu} - \left[\frac{\pi_n(-c)}{\pi_n(c)}\right]^2 \left(\frac{1-c}{1+c}\right)^{\beta-\alpha} \frac{1}{c + x_\nu} \right\} = \lambda_\nu(c)[\pi_n'(x_\nu)]^2 \frac{\mathrm{d}x_\nu}{\mathrm{d}c},$$

and Gautschi has stated the following wondrous conjecture:

Conjecture 1.1.

For any $n \geq 1$, $\alpha > -1$, $\beta > -1$ with $\alpha < \beta$, and for any c with $0 < c \leq 1$, there holds

$$\left[\frac{\pi_n(-c)}{\pi_n(c)}\right]^2 \left(\frac{1-c}{1+c}\right)^{\beta-\alpha} < 1, \tag{3}$$

where $\pi_n(\,\cdot\,) = \pi_n^{(\alpha,\beta)}(\,\cdot\,;c)$ is the subrange Jacobi polynomial of degree n orthogonal on $[-c, c]$ with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$.

Empirical evidence in support of the conjecture is provided in [6, Appendix B].

In this note, for a given $c \in (0, 1)$ we give proof of this conjecture in the domain

$$A(c) = \left\{ (\alpha, \beta) \;\middle|\; \beta \geq \frac{1-c}{1+c}\alpha \;\text{ if }\; -1 < \alpha \leq 0 \right\} \bigcup \left\{ (\alpha, \beta) \;\middle|\; \beta \geq \frac{1+c}{1-c}\alpha \;\text{ if }\; \alpha \geq 0 \right\}. \tag{4}$$

The problem is still open in the domains

$$B(c) = \left\{(\alpha,\beta) \;\Big|\; \alpha \le \beta < \frac{1-c}{1+c}\,\alpha \;\text{ if }\; -1 < \alpha < 0\right\} \tag{5}$$

and

$$C(c) = \left\{(\alpha,\beta) \;\Big|\; \alpha < \beta < \frac{1+c}{1-c}\,\alpha \;\text{ if }\; \alpha \ge 0\right\}. \tag{6}$$

2. Proof of Conjecture (1.1) in the domain $A(c)$

In this section we consider the problem transformed from $[-c,c]$ to $[-1,1]$ by the simple change of variables $x = ct$. The equivalent form of Conjecture (1.1) can be formulated for the weight function on $[-1,1]$, given by

$$W(t) = w(ct) = (1-ct)^{\alpha}(1+ct)^{\beta}, \quad 0 < c < 1,\; -1 < \alpha < \beta,$$

and the corresponding monic orthogonal polynomials $\Pi_n(t)$, given by $\Pi_n(t) = \pi_n(ct)/c^n$, $n \in \mathbb{N}$.

Then the inequality (3) in Conjecture 1.1 becomes

$$\left(\frac{\Pi_n(-1)}{\Pi_n(1)}\right)^2 \frac{W(-1)}{W(1)} < 1, \tag{7}$$

i.e.,

$$D = W(1)\Pi_n(1)^2 - W(-1)\Pi_n(-1)^2 > 0.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(W(t)\Pi_n(t)^2\right) = W'(t)\Pi_n(t)^2 + 2W(t)\Pi_n(t)\Pi_n'(t),$$

we have

$$\begin{aligned} D &= \int_{-1}^{1}\left[W'(t)\Pi_n(t)^2 + 2W(t)\Pi_n(t)\Pi_n'(t)\right]\mathrm{d}t \\ &= \int_{-1}^{1} W'(t)\Pi_n(t)^2\mathrm{d}t + 2\int_{-1}^{1} W(t)\Pi_n(t)\Pi_n'(t)\mathrm{d}t \end{aligned} \tag{8}$$

i.e.,

$$D = \int_{-1}^{1} W'(t)\Pi_n(t)^2\mathrm{d}t,$$

because of the orthogonality, the second integral in (8) is equal to zero.

Since

$$W'(t) = -\alpha c(1-ct)^{\alpha-1}(1+ct)^{\beta} + \beta c(1-ct)^{\alpha}(1+ct)^{\beta-1},$$

i.e.,

$$W'(t) = c\frac{W(t)}{1-c^2t^2}\Phi(ct),$$

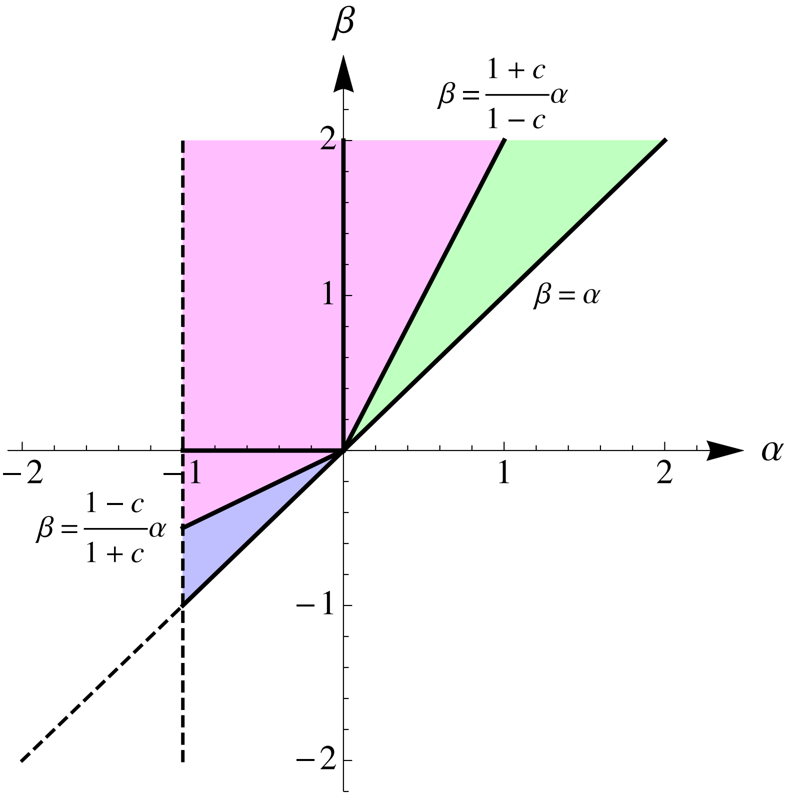
where

$$\Phi(z) = \beta - \alpha - (\beta + \alpha)z$$

and $ct \in (-1, 1)$, we see that

$$\Phi(-1) = 2\beta, \quad \Phi(1) = -2\alpha.$$

Figure 1. **Different domains for** $(\alpha, \beta) \in \mathbb{R}^2$



Evidently, $\Phi(ct) \geq 0$ for each $c \in (0, 1)$ and $t \in (-1, 1)$, if $\beta \geq 0$ and $\alpha \leq 0$, i.e., if

$$(\alpha, \beta) \in \left\{ (\alpha, \beta) \,\middle|\, -1 < \alpha \leq 0, \ \beta \geq 0 \right\}$$

(see Figure 1, colored part in the second quadrant).

However, the conjecture is true if

$$\beta - \alpha - |\beta + \alpha| c \geq 0,$$

that is,

$$c \leq \frac{\beta - \alpha}{|\beta + \alpha|}$$

(as always, $\beta - \alpha > 0$). If the right-hand side is ≥ 1, the conjecture is true unrestrictedly, for all $0 < c < 1$. This is the case, as we mentioned before, if $\beta \geq 0$ and $\alpha \leq 0$ (colored part in the second quadrant).

In the square $-1 < \alpha < 0,\ -1 < \beta < 0$, the conjecture is true if

$$c \leq \frac{|\alpha| - |\beta|}{|\alpha| + |\beta|} = \frac{\alpha - \beta}{\alpha + \beta},$$

i.e., when $\beta > \alpha(1-c)/(1+c)$.

In the domain $\alpha > 0,\ \beta > 0,\ \beta > \alpha$, (7) holds if

$$c \leq \frac{\beta - \alpha}{\beta + \alpha},$$

i.e., when $\beta > \alpha(1+c)/(1-c)$.

That summarizes the current state of the conjecture.

Theorem 2.1.
For each c with $0 < c \leq 1$, the inequality (7), i.e., (3), holds if $(\alpha, \beta) \in A(c)$, where $A(c)$ is a domain in $\mathbb{R}^2$ defined by (4).

Thus, the conjecture is true in the domain in $\mathbb{R}^2$, which is colored magenta in Figure 1.

The problem is still open in the domains $B(c)$ and $C(c)$, given by (5) and (6), respectively. These domains are colored in blue and green in Figure 1.

Remark 2.1.
For orthogonal polynomials and Gaussian quadratures see books [4] and [9]. For numerical and symbolic constructions of orthogonal polynomials and Gaussian quadrature formulas there is a software package in Mathematica (see [1] and [10]).

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