On a weighted version of the Gumbel-Barnett copula

Research Article

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Abstract: Copulas are increasingly widely used probabilistic tools for describing, analyzing, and modeling random variable dependencies. In this article, we offer a new copula which stands out from the others by an original definition based on the simple symmetric two-dimensional function "$x^y y^x$" multiplied with an exponential function. It can also be viewed as a special weighted version of the Gumbel-Barnett copula. The following properties are demonstrated: the new copula extends the independence copula; it is symmetric, not Archimedean, and not radially symmetric; it has no tail dependence; and it is ideal to model weak positive correlations, as shown by the investigation of the medial, Spearman, and Kendall correlations. Some graphical and numerical analyses are also provided to illustrate the findings.

MSC: 62H99

Keywords: Probability ● Copula ● Power function ● Multivariate distributions ● Dependence

1.   Introduction

Copulas can be viewed as functions that create multivariate distributions by joining uniform marginal distributions. On the other hand, they are useful in separating joint distributions into marginal distributions and assessing the interdependence of the underlying random variables. In the two-dimensional case, the definition of a copula is provided below.

Definition 1.1.
A two-dimensional copula is a cumulative distribution function on $[0,1]^2$ with standard uniform marginal distributions. In particular, in the absolutely continuous case, the function $C : [0,1]^2 \to [0,1]$ is a two-dimensional copula if and only if

- $C(x,0) = C(0,y) = 0$ for any $(x,y) \in [0,1]^2$,

- $C(x,1) = x$ and $C(1,y) = y$ for any $(x,y) \in [0,1]^2$,

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- $\partial^2 C(x,y)/(\partial x \partial y) \geq 0$ for any $(x,y) \in [0,1]^2$.

The books [8], [5], and [7] contain all the required information regarding copulas. Naturally, there are many types of copulas with different shapes, allowing copula-based modeling to be adaptable. Among the classical copulas (or family of copulas), we may cite the Frank, Gumbel-Hougaard, Clayton, Ali-Mikhail-Haq, Joe, Farlie-Gumbel-Morgenstern, Plackett, Raftery, Galambos, Hüsler-Reiss, elliptical, Fréchet, Marshall-Olkin and Gumbel-Barnett (GB) copulas. Modern applications of the standard copulas can be found in [10], [9], [15], and [11], and recent developments are available in [12], [13], [14], [2], [3], and [4].

For the purpose of this study, let us briefly present the GB copula. To begin, the GB copula is the function $C_o : [0,1]^2 \to [0,1]$ defined by

$$C_o(x,y) = xy e^{-\theta \log(x) \log(y)},$$

with $\theta \in [0,1]$, and the convention-notation $\log(0) = \lim_{x \to 0} \log(x) = -\infty$. It is widely credited to [1], and is known to be an Archimedean copula, usually used to model weak and not necessarily linear correlations. For the theoretical properties and applications of the GB copula, we may refer to [16] and [6], and the references therein. In this article, attention is paid to the simple two-dimensional function "$x^y y^x$", and its possible use in a copula context. On the mathematical side, the function "$x^y y^x$" is fascinating thanks to its symmetrical and power-variable structure. As a first result, we show that it is not a valid copula, which is not immediately apparent at first glance. Then, we introduce and investigate a new type of two-dimensional copula based on this function. We thus treat the mathematical challenge to propose modifications of "$x^y y^x$" to make it a valid copula. To the best of our knowledge, this remains an unexamined direction in the literature. To this end, a special weighted version of the GB copula is proposed and studied. Then, a more general parametric version of this copula with one parameter is developed. Its main features are determined. The relationships of the new copula with other well-known copulas are addressed, and some important dependent indices are evaluated. The results are supported by graphics and numerical tables.

The paper is divided into the following sections: Section 2 gives the first results on the research about two-dimensional copulas defined with "$x^y y^x$". Section 3 is devoted to the main proposed copula, with its qualities. A conclusion section is provided in Section 4.

2. First results

We begin by studying the possible copula nature of the two-dimensional function "$x^y y^x$".

Proposition 2.1.
Let us consider the function $\Xi : [0,1]^2 \to [0,1]$ *defined by*

$$\Xi(x,y) = x^y y^x.$$

Then $\Xi(x, y)$ is not a valid two-dimensional copula.

Proof.   Let us check if the required properties defining a copula hold.

- For any $x \in (0, 1]$, we have $\Xi(x, 0) = x^0 0^x = 0$, and, similarly, for any $y \in (0, 1]$, $\Xi(0, y) = 0$. The case $(x, y) = (0, 0)$ is special; with the convention $0^0 = 1$, we obtain $\Xi(x, y) = 0^0 0^0 = 1 \neq 0$. We can however define $\Xi(x, y)$ by assuming that $\Xi(0, 0) = 0$, which can be considered as a correction of continuity.

- For any $x \in [0, 1]$, we have $\Xi(x, 1) = x^1 1^x = x$ and, similarly, for any $y \in [0, 1]$, $\Xi(1, y) = y$.

- For any $(x, y) \in [0, 1]^2$, using standard derivation techniques, we obtain

$$\frac{\partial^2}{\partial x \partial y} \Xi(x, y) = x^{y-1} y^{x-1} \phi(x, y),$$

where

$$\phi(x, y) = x^2 \log(y) + xy + y \log(x)(x \log(y) + y) + x + y.$$

It is clear that $x^{y-1} y^{x-1} \geq 0$. On the other hand, we have, for example, $\phi(e^{-2}, 1) = 2e^{-2} - 1 < 0$, implying that we do not have $\partial^2 \Xi(x, y)/(\partial x \partial y) \geq 0$ for any $(x, y) \in [0, 1]^2$.

As a result, $\Xi(x, y)$ is not a valid two-dimensional copula. This ends the proof of Proposition 2.1.   □

Thus, Proposition 2.1 states that "$x^y y^x$" is not a copula. With the construction of the GB copula in mind, the next result presents a modification of "$x^y y^x$" to make it a valid two-dimensional copula.

Proposition 2.2.
Let us consider the function $C_* : [0, 1]^2 \to [0, 1]$ defined by

$$C_*(x, y) = x^y y^x e^{-\log(x)\log(y)},$$

with the convention $0^0 = 1$, and the convention-notation $\log(0) = \lim_{x \to 0} \log(x) = -\infty$. Then $C_*(x, y)$ is a valid two-dimensional copula.

Proof.   Let us see if the properties that define a copula hold.

- For any $x \in [0, 1]$, thanks to the exponential term, we have $C_*(x, 0) = \lim_{y \to 0} x^y y^x e^{-\log(x)\log(y)} = 0$, and, for any $y \in [0, 1]$, $C_*(0, y) = \lim_{x \to 0} x^y y^x e^{-\log(x)\log(y)} = 0$.

- For any $x \in (0, 1]$, we have $C_*(x, 1) = x^1 1^x e^{-\log(x)\log(1)} = x$ and $C_*(0, 1) = 0$ and, similarly, for any $y \in [0, 1]$, $C_*(1, y) = y$.

- For any $(x, y) \in [0, 1]^2$, using standard derivation techniques, we establish that

$$\frac{\partial^2}{\partial x \partial y} C_*(x, y) = y^{x-1} x^{y - \log(y) - 1} \psi(x, y),$$

where

$$\psi(x, y) = xy + (x - 1)x \log(y) + (y - 1) \log(x)((x - 1) \log(y) + y) + x + y - 1.$$

It is clear that $y^{x-1} x^{y - \log(y) - 1} \geq 0$. Let us now study the sign of $\psi(x, y)$. It follows from the inequality: $\log(x) \leq x - 1$ for any $x > 0$, and $(x, y) \in [0, 1]^2$, and appropriate factorizations, that

$$\psi(x, y) \geq xy + (x - 1)x(y - 1) + (y - 1)(x - 1)((x - 1)(y - 1) + y) + x + y - 1$$

$$= xy[2 + (1 - x)(1 - y)] \geq 0.$$

Therefore, $\partial^2 C_*(x, y)/(\partial x \partial y) \geq 0$ for any $(x, y) \in [0, 1]^2$.

As a result, $C_*(x, y)$ is a valid two-dimensional copula. This ends the proof of Proposition 2.2. □

Remark 2.1.
It is worth noting that the convention $0^0 = 1$ does not play a role in the proof of Proposition 2.2. In particular, we can consider the sometimes adopted convention $0^0 = 0$ without change in the proof.

To the best of our knowledge, the copula $C_*(x, y)$ is new in the literature. Since

$$C_*(x, y) = x^{y-1} y^{x-1} C_o(x, y),$$

it can be viewed as a weighted version of the GB copula. At a first glance, a drawback of this copula is that it is not flexible; there is no parameter to modulate its characteristics. For this reason, an extended one-parameter version of this copula is developed and studied in the next section.

3. A new parametric copula

The main copula of the study, as well as its critical features, are discussed in this section.

3.1. Definition

The proposed parametric copula is expressed in the next proposition.

Proposition 3.1.
Let us consider the function $C_\star : [0, 1]^2 \to [0, 1]$ defined by

$$C_\star(x, y) = x^{y^\theta} y^{x^\theta} e^{-\theta \log(x) \log(y)},$$

with $\theta \in [0, 1]$, the convention $0^0 = 1$, and convention-notation $\log(0) = \lim_{x \to 0} \log(x) = -\infty$. Then $C_\star(x, y)$ is a valid two-dimensional copula.

Proof. Let us see if the properties that define a copula are still valid.

- For any $x \in [0,1]$, we have $C_\star(x,0) = \lim_{y \to 0} x^{y^\theta} y^{x^\theta} e^{-\theta \log(x) \log(y)} = 0$, and, for any $y \in [0,1]$, $C_\star(0,y) = \lim_{x \to 0} x^{y^\theta} y^{x^\theta} e^{-\theta \log(x) \log(y)} = 0$.

- For any $x \in (0,1]$, we have $C_\star(x,1) = x^{1^\theta} 1^{x^\theta} e^{-\theta \log(x) \log(1)} = x$ and $C_\star(0,1) = 0$ and, similarly, for any $y \in [0,1]$, $C_\star(1,y) = y$.

- For any $(x,y) \in [0,1]^2$, using standard derivation techniques, we find that

$$\frac{\partial^2}{\partial x \partial y} C_\star(x,y) = y^{x^\theta - 1} x^{y^\theta - \theta \log(y) - 1} \Lambda(x,y),$$

where

$$\Lambda(x,y) = x^\theta y^\theta + \theta \log(x)(y^\theta - 1)\left[\theta(x^\theta - 1)\log(y) + y^\theta\right] + \theta(x^\theta - 1)x^\theta \log(y) + \theta x^\theta + \theta y^\theta - \theta.$$

Clearly, we have $y^{x^\theta - 1} x^{y^\theta - \theta \log(y) - 1} \geq 0$. Let us now study the sign of $\Lambda(x,y)$. It follows from the inequality: $\log(x) \leq x - 1$ for any $x > 0$, so $\theta \log(x) = \log(x^\theta) \leq x^\theta - 1$, and appropriate factorizations, that

$$\begin{aligned}
\Lambda(x,y) &\geq x^\theta y^\theta + (x^\theta - 1)(y^\theta - 1)\left[(x^\theta - 1)(y^\theta - 1) + y^\theta\right] \\
&\quad + (x^\theta - 1)x^\theta(y^\theta - 1) + \theta x^\theta + \theta y^\theta - \theta \\
&= x^\theta y^\theta \left[1 + \theta + (1 - x^\theta)(1 - y^\theta)\right] + (1 - \theta)(1 - x^\theta)(1 - y^\theta).
\end{aligned}$$

Since $(x,y,\theta) \in [0,1]^3$, we have $\Lambda(x,y) \geq 0$, implying that $\partial^2 C_\star(x,y)/(\partial x \partial y) \geq 0$ for any $(x,y) \in [0,1]^2$.

As a result, $C_\star(x,y)$ is a valid two-dimensional copula. This ends the proof of Proposition 3.1. $\qquad\square$

Since

$$C_\star(x,y) = x^{y^\theta - 1} y^{x^\theta - 1} C_o(x,y),$$

this copula can be viewed as a one-parameter weighted version of the GB copula, with a weight function depending on the parameter θ. We thus call it the parametric weighted GB copula, abbreviated as "PWGB copula" for short. Figures 1, 2 and 3 display the perspective and contour plots of this copula for several values of θ.
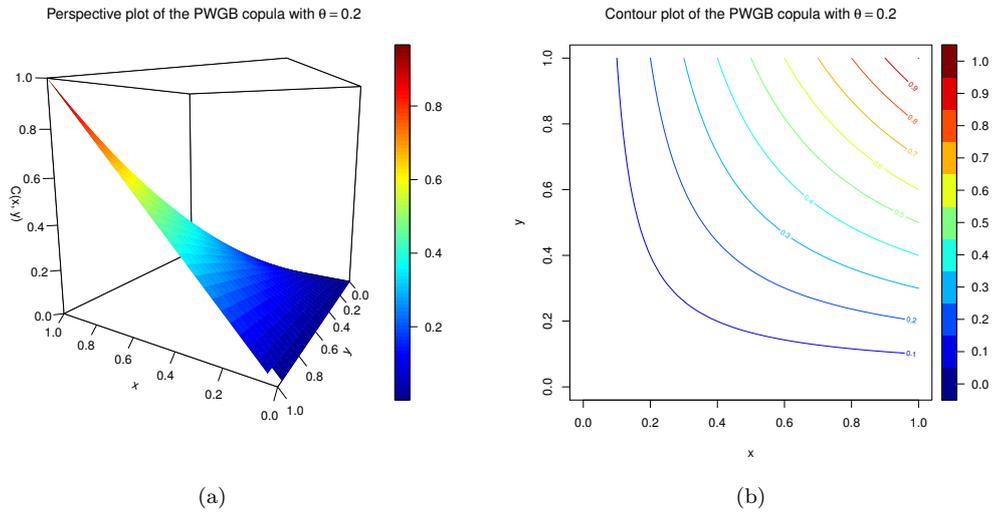
Perspective plot of the PWGB copula with θ = 0.2

Contour plot of the PWGB copula with θ = 0.2

(a)

(b)

Figure 1. Representations of the (a) perspective plot and (b) contour plot of the PWGB copula for $\theta = 0.2$

Perspective plot of the PWGB copula with θ = 0.5

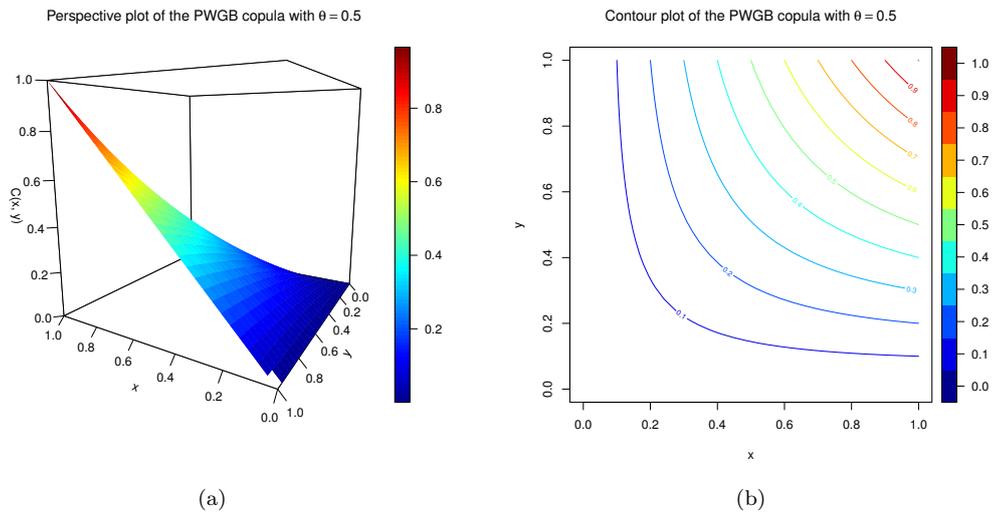Contour plot of the PWGB copula with θ = 0.5

(a)

(b)

Figure 2. Representations of the (a) perspective plot and (b) contour plot of the PWGB copula for $\theta = 0.5$
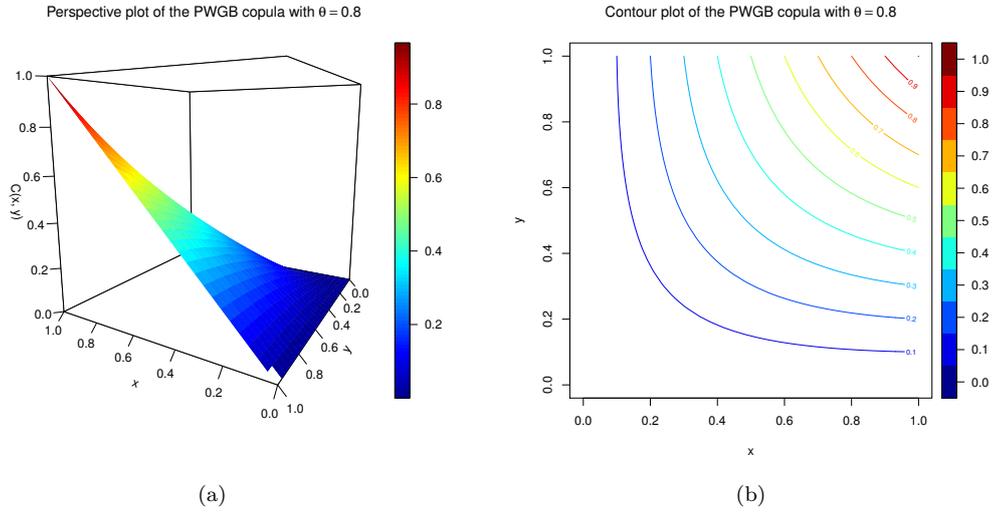
Figure 3. Representations of the (a) perspective plot and (b) contour plot of the PWGB copula for $\theta = 0.8$

Some functions related to the PWGB copula are presented below.

Based on the PWGB copula, the PWGB copula density is the function $c_\star : [0,1]^2 \to [0,+\infty)$ given by

$$c_\star(x,y) = \frac{\partial^2}{\partial x \partial y} C_\star(x,y) = y^{x^\theta - 1} x^{y^\theta - \theta \log(y) - 1} \times$$
$$\left\{ x^\theta y^\theta + \theta \log(x)(y^\theta - 1) \left[\theta(x^\theta - 1)\log(y) + y^\theta \right] + \theta(x^\theta - 1)x^\theta \log(y) + \theta x^\theta + \theta y^\theta - \theta \right\}.$$

The possible shapes of this function characterizes the modeling capability of the PWGB copula model. Figures 4, 5 and 6 present the perspective and contour plots of this copula density for several values of θ.
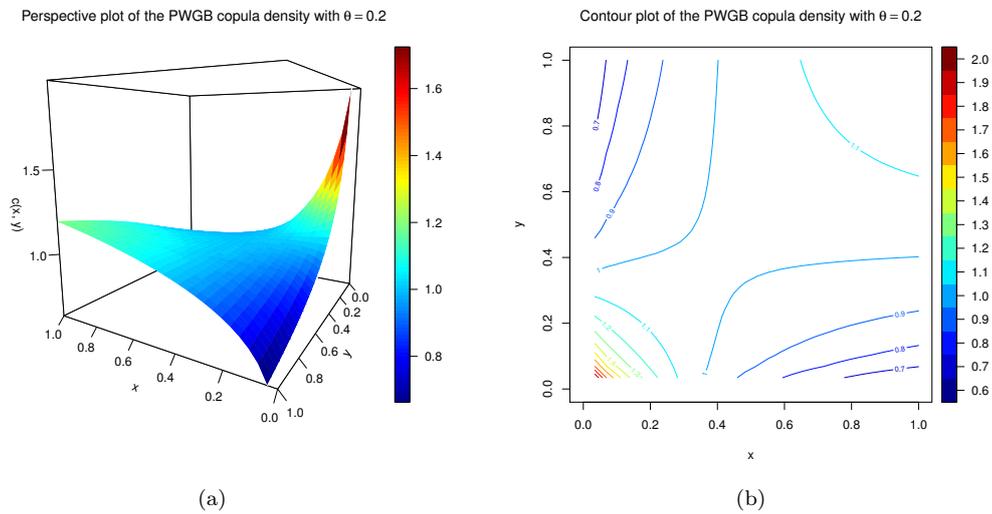

Figure 4. Representations of the (a) perspective plot and (b) contour plot of the PWGB copula density for $\theta = 0.2$
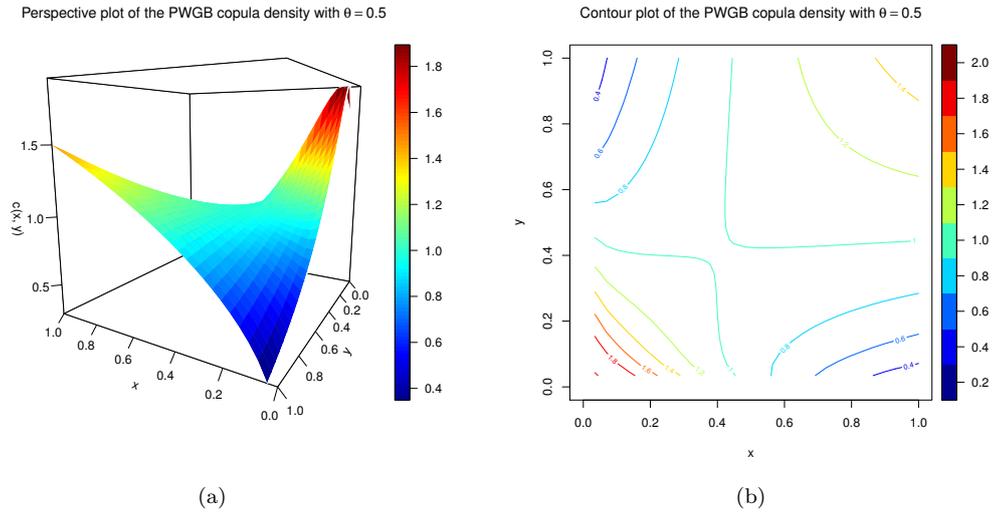
Figure 5. Representations of the (a) perspective plot and (b) contour plot of the PWGB copula density for $\theta = 0.5$
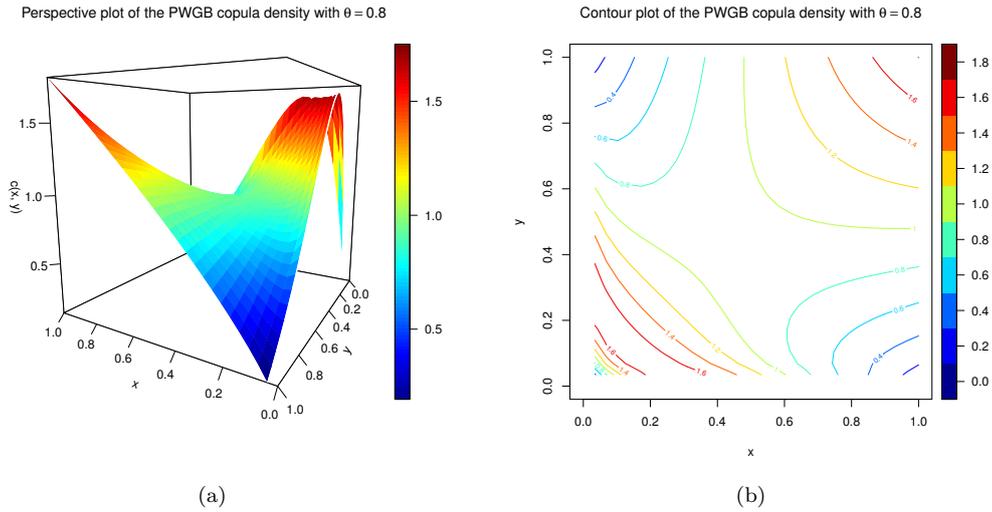

Figure 6. Representations of the (a) perspective plot and (b) contour plot of the PWGB copula density for $\theta = 0.8$

From these figures, we see how the parameter θ affects the shapes of the PWGB copula density. It is clear that it mainly affects the tails and skewness of this function. Thus, thanks to the parameter θ, the PWGB copula is enough flexible to be used in various modeling contexts.

The PWGB copula density also plays a crucial role in the determination of moment-type measures, and estimation methods. In particular, the parameter θ can be estimated from n observations $(x_1, y_1), \ldots, (x_n, y_n)$ coming from the PWGB copula distribution by the maximum likelihood method; θ is thus estimated by $\widetilde{\theta}$, where

$$\widetilde{\theta} = \operatorname{argmax}_\theta \prod_{i=1}^{n} c_\star(x_i, y_i).$$

As a last important function, the survival PWGB copula is the function $\widehat{C}_\star : [0,1]^2 \to [0,1]$ defined by

$$\widehat{C}_\star(x,y) = x + y - 1 + C_\star(1-x, 1-y)$$
$$= x + y - 1 + (1-x)^{(1-y)^\theta}(1-y)^{(1-x)^\theta}e^{-\theta \log(1-x)\log(1-y)}.$$

It defines a valid copula, which is also new in the literature.

Because any convex linear combination of copulas is a copula (see [8]), the PWGB copula can be used in a variety of new copula constructions. For instance, we can consider the mixed copula $C_\nabla : [0,1]^2 \to [0,1]$ defined by

$$C_\nabla(x,y) = \xi C_\star(x,y) + (1-\xi)C_o(x,y)$$
$$= xye^{-\theta \log(x)\log(y)}\left[\xi x^{y^\theta-1}y^{x^\theta-1} + 1 - \xi\right],$$

with $\xi \in [0,1]$; the GB copula is obtained for $\xi = 0$, whereas the PWGB copula is obtained for $\xi = 1$, all the intermediary values for ξ providing a weighted compromise between these two copulas. As another example, we can consider the mixed copula $C_\triangle : [0,1]^2 \to [0,1]$ defined by

$$C_\triangle(x,y) = \frac{1}{2}\left[C_\star(x,y) + y - C_\star(1-x,y)\right]$$
$$= \frac{1}{2}\left[x^{y^\theta}y^{x^\theta}e^{-\theta \log(x)\log(y)} + y - (1-x)^{y^\theta}y^{(1-x)^\theta}e^{-\theta \log(1-x)\log(y)}\right].$$

The copulas above are also brand new and comparable to the PWGB copula in terms of functionality.

3.2. Properties

We now list some important properties of the PWGB copula.

- For $\theta = 0$, we have $C_\star(x,y) = xy$; the PWGB copula is reduced to the independence copula.

- We can relate the PWGB copula and the copula introduced in Proposition 2.2 by the following relationship: $C_\star(x,y) = C_*(x^\theta, y^\theta)^{1/\theta}$. In particular, for $\theta = 1$, we have $C_\star(x,y) = C_*(x,y)$.

- The PWGB copula is symmetric since $C_\star(x,y) = C_\star(y,x)$ for any $(x,y) \in [0,1]^2$.

- The PWGB copula can be expressed under the following simple exponential form:

$$C_\star(x,y) = e^{\left[x^\theta y^\theta - (\theta \log(x)-x^\theta)(\theta \log(y)-y^\theta)\right]/\theta}.$$

Since separable functions in terms of x and y are involved, this expression can serve as a first result to obtain a higher dimensional version of the PWGB copula.

- One can remark that, for $x = 1/4$, $y = 1/2$ and $z = 1/3$ arbitrary chosen, we have

$$C_\star\left(x, C_\star\left(y, z\right)\right) = 0.05963137 \neq 0.0612156 = C_\star\left(C_\star\left(x, y\right), z\right).$$

This implies that the PWGB copula is not Archimedean for $\theta = 1$ (see [8]). The same can be proved for $\theta \in (0, 1]$.

- The PWGB copula is not radially symmetric since there exists (x, y) such that $\widehat{C}_\star(x, y) \neq C_\star(x, y)$.

- The following copula order holds: For any $(x, y) \in [0, 1]^2$, we have $C_\star(x, y) \geq C_o(x, y)$.

- The Fréchet-Hoeffding bounds hold: For any $(x, y) \in [0, 1]^2$, we have $\max(x + y - 1, 0) \leq C_\star(x, y) \leq \min(x, y)$.

- For any $\theta \in [0, 1]$, we have

$$\lambda_L = \lim_{x \to 0} \frac{C_\star(x, x)}{x} = \lim_{x \to 0} x^{2x^\theta - 1} e^{-\theta(\log(x))^2} = 0$$

and

$$\lambda_U = \lim_{x \to 1} \frac{1 - 2x + C_\star(x, x)}{1 - x} = \lim_{x \to 1} \frac{1 - 2x + x^{2x^\theta} e^{-\theta(\log(x))^2}}{1 - x} = 0.$$

As a result, the PWGB copula has no tail dependence. Further detail on this notions can be found in [8].

- The medial correlation of the PWGB copula is defined by

$$M = 4C_\star\left(\frac{1}{2}, \frac{1}{2}\right) - 1.$$

Thus, it can be expressed as

$$M = 2^{2 - 2^{1-\theta}} e^{-\theta(\log(2))^2} - 1.$$

Table 1 determines its numerical values for $\theta = 0, 0.1, 0.2, \dots, 1$.

Table 1. Values of the medial correlation of the PWGB copula for several values of θ

θ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
M	0	0.0458	0.0869	0.1231	0.1543	0.1803	0.2013	0.2172	0.2284	0.2348	0.2370

It is worth noting that $M \in [0, 0.24)$

- A useful dependence measure based on copula is the Spearman rho (see [8]). The Spearman rho of the PWGB copula is defined by

$$\rho = 12 \int_0^1 \int_0^1 C_\star(x, y) dx dy - 3.$$

It has not a closed-form. We, however, propose a table for the numerical values of ρ in Table 2 for $\theta = 0, 0.1, 0.2, \ldots, 1$.

Table 2. Values of the Spearman rho of the PWGB copula for several values of θ

θ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ρ	0	0.0708	0.1317	0.1814	0.2200	0.2480	0.2667	0.2773	0.2812	0.2794	0.2729

We can observe that ρ is not a monotonic function with respect to θ, and that $\rho \in [0, 0.3)$. Thus, as for the GB copula, the PWGB copula can be used to model weak correlations.

- In complement of the Spearman rho, we can present the Kendall tau of the PWGB copula. It is defined by

$$\tau = 4 \int_0^1 \int_0^1 C_\star(x, y) c_\star(x, y) dx dy - 1.$$

Like the Spearman rho, this integral measure has not a closed-form. Table 3 proposes a table for the numerical values of τ for $\theta = 0, 0.1, 0.2, \ldots, 1$.

Table 3. Values of the Kendall tau of the PWGB copula for several values of θ

θ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
τ	0	0.0472	0.0879	0.1211	0.1465	0.1646	0.1762	0.1821	0.1834	0.1806	0.1746

With respect to θ, we can see that τ is not a monotonic function, and that $\tau \in [0, 0.19)$. The small values of τ confirm the fact that the PWGB copula is ideal to model weak correlations.

- Last but not least, by considering two unidimensional cumulative distribution functions, say $F(x)$ and $G(x)$, we define a new two-dimensional distribution by the cumulative distribution function $H : \mathbb{R}^2 \to [0, 1]$ given as

$$H(x, y) = C_\star(F(x), G(y)) = F(x)^{G(y)^\theta} G(y)^{F(x)^\theta} e^{-\theta \log[F(x)] \log[G(y)]}.$$

Based on this definition, the possibilities of new two-dimensional distributions are thus infinite.

4. Conclusion

In this article, we have developed a new parametric two-dimensional copula based on the function "$x^y y^x$" multiplied with an exponential function. It can be viewed as a special extension of the Gumbel-Barnett copula. We have highlighted the numerous qualities of this new copula and presented some relationships with well-known copulas. This study is thus useful since it adds to the literature's copula repertory. The newly created copula is believed to be beneficial in modeling real data sets by practitioners, which remains a direction of work for the future.

Competing Interests

The author has no conflicts of interest to declare

Acknowledgements

The author thanks the referees for the thorough comments on the paper.

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