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Restriction method for approximating cube roots

Research Article

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Abstract: The aim of this paper is to present a new method called "Restriction Method for Approximating Cube Roots", which helps students to find an approximate value for any cube root of any rational number easily and simply. Also, we prove this method and we give some examples that enhance our method.

Keywords: Full Cube Number • Successive Full Cube Numbers

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1. Introduction

Many methods were presented to find approximate values for cube roots of rational numbers [2, 6]. These methods were built based on functions, sequences, series or derivatives…etc.; which are basic assumptions to other methods [1, 5]. The presented method of finding approximate values for cube roots is very simple to be applied by students in the preliminary stages. In particular, our method can be applied easily in comparison with other methods utilized for the same purpose.

2. Restriction Method for Approximating Cube Roots

At the beginning of this section, we introduce some definitions that will be used to present our method and its proof.

Definition 2.1.
Let $a \in \mathbb{N}$, then a^3 is called a Full Cube Number since $a^3 = a * a * a$.

e.g. 64 is a Full Cube Number for 4 since $64 = 4 * 4 * 4$.

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Definition 2.2.

a^3, b^3 are two Successive Full Cube Numbers $\Leftrightarrow$ $\exists\, a, b$ such that: $a, b \in \mathbb{N}$ and $b = a + 1$ respectively.

e.g. $27, 64$ are two Successive Full Cube Numbers for $3, 4$ respectively [3, 4].

Corollary 2.3 (New result).

Let $x \in \mathbb{R}$ and $a, b \in \mathbb{N}$ such that $b = a + 1$ and $a^3 < x < b^3$, then:

$$\sqrt[3]{x} \approx \frac{4x + a^3 + b^3 + 2(a^2 + b^2) - 1 + (a + b)(x - a^3 + \frac{a + b + 1}{2})}{2(a^2 + b^2 + ab + x - a^3 + 1) + 3(a + b)} \tag{1}$$

Proof. Since $a^3 < x < b^3$, then $a < \sqrt[3]{x} < b$. So; we can assume that:

$$\sqrt[3]{x} \approx \frac{a + b}{2} \tag{2}$$

is the first approximation for $\sqrt[3]{x}$.

Thus;

$$x \approx \frac{(a + b)^3}{8}$$

or

$$\sqrt[3]{x} \approx \frac{a + b}{2} \approx \frac{4x}{(a + b)^2} \tag{3}$$

Now, since

$$b - a = 1 \Rightarrow (b - a)^2 = 1$$

$$b^2 + a^2 = 2ab + 1 \tag{4}$$

$$b^2 + a^2 = 2(ab + \frac{1}{2})$$

$$\Rightarrow \frac{a^2 + b^2}{2} = ab + \frac{1}{2}$$

or

$$a^2 - \frac{a^2}{2} + b^2 - \frac{b^2}{2} = ab + \frac{1}{2}$$

$$a^2 + b^2 - ab = \frac{a^2 + b^2 + 1}{2} \tag{5}$$

By multiplying Eq.(5) by $(a+b)$, we get:

$$(a+b)(a^2-ab+b^2)=\frac{(a+b)(a^2+b^2+1)}{2}$$

$$\Rightarrow a^3+b^3=\frac{(a+b)(a^2+b^2+1)}{2}$$

or

$$\frac{a+b}{2}=\frac{a^3+b^3}{a^2+b^2+1} \tag{6}$$

Also, by adding (a^2+b^2) to Eq.(4) we get:

$$2a^2+2b^2=a^2+2ab+b^2+1 \Rightarrow 2(a^2+b^2)=(a+b)^2+1$$

$$\Rightarrow a^2+b^2=\frac{(a+b)^2}{2}+\frac{1}{2} \Rightarrow a^2+b^2=(a+b)\left[\frac{a+b}{2}+\frac{1}{2(a+b)}\right]$$

$$\Rightarrow \frac{a^2+b^2}{a+b}=\frac{a+b}{2}+\frac{1}{2(a+b)} \Rightarrow \frac{a+b}{2}=\frac{a^2+b^2}{a+b}-\frac{1}{2(a+b)}$$

$$\frac{a+b}{2}=\frac{2a^2+2b^2-1}{2(a+b)} \tag{7}$$

Moreover, by adding (ab) to Eq.(4) we get:

$$b^2+ab+a^2=1+3ab \tag{8}$$

By multiplying Eq.(8) by $(b-a)$, we get:

$$(b-a)(b^2+ab+a^2)=(1+3ab)(b-a) \Rightarrow b^3-a^3=(b-a)(1+3ab)$$

and since $(b-a)=1$, then

$$b^3-a^3=(1+3ab) \tag{9}$$

By adding and subtracting x to the right hand side, we get:

$$b^3-x+x-a^3=1+3ab \tag{10}$$

Also, by adding $(\frac{a+b+1}{2})$ to Eq.(10), we get:

$$b^3-x+x-a^3+\frac{a+b+1}{2}=1+3ab+\frac{a+b+1}{2} \tag{11}$$

Now, by dividing Eq.(11) on $(x-a^3+\frac{a+b+1}{2})$, we get

$$\frac{b^3 - x}{x - a^3 + \frac{a+b+1}{2}} + 1 = \frac{1 + 3ab + \frac{a+b+1}{2}}{x - a^3 + \frac{a+b+1}{2}}$$

$$\Rightarrow \frac{b^3 - x - (1 + 3ab + \frac{a+b+1}{2})}{x - a^3 + \frac{a+b+1}{2}} = -1$$

or

$$\frac{b^3 - x - (1 + 3ab + \frac{a+b+1}{2})}{x - a^3 + \frac{a+b+1}{2}} = -\frac{a+b}{a+b}$$

$$\Rightarrow \frac{(a+b)(b^3 - x - 1 - 3ab - \frac{a+b+1}{2})}{x - a^3 + \frac{a+b+1}{2}} = -(a+b)$$

So;

$$\frac{a+b}{2} = \frac{(a+b)(x - b^3 + 1 + 3ab + \frac{a+b+1}{2})}{2x - 2a^3 + a + b + 1} \tag{12}$$

Now, by applying (3), (6), (7) and (12) on the following Rational Rule:

$$\left(\text{If } \frac{A}{B} = r \text{ and } \frac{C}{D} = r \text{ then } \frac{A+C}{B+D} = r, \text{ where } A, B, C, D, r \in R \right)$$

we get:

$$\sqrt[3]{x} \approx \frac{a+b}{2} = \frac{4x + a^3 + b^3 + 2a^2 + 2b^2 - 1 + (a+b)(x - b^3 + 1 + 3ab + \frac{a+b+1}{2})}{(a+b)^2 + a^2 + b^2 + 1 + 2(a+b) + 2x - 2a^3 + a + b + 1}$$

But, from Eq. (9), we obtain:

$$\sqrt[3]{x} \approx \frac{4x + a^3 + b^3 + 2(a^2 + b^2) - 1 + (a+b)(x - b^3 + b^3 - a^3 + \frac{a+b+1}{2})}{a^2 + 2ab + b^2 + a^2 + b^2 + 1 + 2a + 2b + 2x - 2a^3 + a + b + 1}$$

$$\sqrt[3]{x} \approx \frac{4x + a^3 + b^3 + 2(a^2 + b^2) - 1 + (a+b)(x - a^3 + \frac{a+b+1}{2})}{2a^2 + 2b^2 + 2ab + 3a + 3b + 2(x - a^3) + 2}$$

So, $$\sqrt[3]{x} \approx \frac{4x + a^3 + b^3 + 2(a^2+b^2) - 1 + (a+b)(x - a^3 + \frac{a+b+1}{2})}{2(a^2 + b^2 + ab + x - a^3 + 1) + 3(a+b)}$$

■

Thus, Eq.(1) finds an approximate value for $\sqrt[3]{x}$, where $a + b \in \mathbb{N}$, $x \in \mathbb{R}$ such that $b = a + 1$ and $a^3 < x < b^3$.

Example 2.4.

Use the Restriction Method to find an approximate value for $\sqrt[3]{19}$.

Sol. Since $8 < 19 < 27$, then $2 < \sqrt[3]{19} < 3$. So; here $a = 2, a^2 = 4, a^3 = 8$ and $b = 3, b^2 = 9, b^3 = 27$.

Now, by applying on Eq. (1), we get:

$$\sqrt[3]{19} \approx \frac{4(19) + 8 + 27 + 2(4 + 9) - 1 + (2 + 3)(19 - 8 + \frac{2 + 3 + 1}{2})}{2(4 + 9 + (2)(3) + 19 - 8 + 1) + 3(2 + 3)}$$

$$\sqrt[3]{19} \approx 2.6753$$

We note that (2.6753) is approximately close to the calculated value (2.6684) for $\sqrt[3]{19}$ rounded to four decimal places, and the absolute value of the error between the approximate value and the calculated value is (6.9×10^{-3}).

Example 2.5.

Use the Restriction Method to find an approximate value for $\sqrt[3]{169}$.

Sol.

Since $125 < 169 < 216$, then $5 < \sqrt[3]{169} < 6$. So; $a = 5, a^2 = 25, a^3 = 125$ and $b = 6, b^2 = 36, b^3 = 216$.

Hence, by applying on Eq. (1), we get:

$$\sqrt[3]{169} \approx \frac{4(169) + 125 + 216 + 2(25 + 36) - 1 + (5 + 6)(169 - 125 + \frac{5 + 6 + 1}{2})}{2(25 + 36 + (5)(6) + 169 - 125 + 1) + 3(5 + 6)}$$

$$\sqrt[3]{169} \approx 5.5344$$

(5.5344) is approximately close to the calculated value (5.5288) for $\sqrt[3]{169}$ rounded to four decimal places, and the absolute value of the error between the approximate value and the calculated value is (5.6×10^{-3}).

Example 2.6.

Use the Restriction Method to find an approximate value for $\sqrt[3]{503}$.

Sol. Since $343 < 503 < 512$, then $7 < \sqrt[3]{503} < 8$. So; $a = 7, a^2 = 49, a^3 = 343$ and $b = 8, b^2 = 64, b^3 = 512$.

By applying on Eq. (1), we get:

$$\sqrt[3]{503} \approx \frac{4(503) + 343 + 512 + 2(49 + 64) - 1 + (7 + 8)\left(503 - 343 + \frac{7 + 8 + 1}{2}\right)}{2(49 + 64 + (7)(8) + 503 - 343 + 1) + 3(7 + 8)}$$

$$\sqrt[3]{503} \approx 7.9603$$

Also, (7.9603) is approximately close to the calculated value (7.9528) for $\sqrt[3]{503}$ rounded to four decimal places, and the absolute value of the error between the approximate value and the calculated value is (7.5×10^{-3}).

3. Using Restriction Method to Approximate Some Cube Roots

The following table shows some examples that demonstrate the Restriction Method to approximate some of the cube roots for random numbers.

Table 1. Approximating some cube roots for randomly chosen numbers using the Restriction Method.

$\sqrt[3]{x}$	The approximate value using Restriction Method	The calculated value	The absolute value of the error
$\sqrt[3]{3}$	1.4483	1.4422	0.0060
$\sqrt[3]{14}$	2.4030	2.4101	0.0071
$\sqrt[3]{26}$	2.9560	2.9625	0.0065
$\sqrt[3]{39}$	3.3719	3.3912	0.0193
$\sqrt[3]{57}$	3.8599	3.8485	0.0114
$\sqrt[3]{89}$	4.4577	4.4647	0.0070
$\sqrt[3]{171}$	5.5599	5.5505	0.0094
$\sqrt[3]{427}$	7.3571	7.5302	0.0069
$\sqrt[3]{992.5}$	9.9792	9.9749	0.0043
$\sqrt[3]{1178.7}$	10.5777	10.5633	0.0144

4. Comparing Restriction Method for Approximating Cube Roots with Other Method

In this section, we will compare our method with some other methods that show the results which are very close to our method results for some cube roots, these methods are Newton-Raphson Method, Binomial Series, Linear Approximation, and Taylor's Polynomial respectively.

The following table shows the efficiency of each method, it also shows the fourth approximation for the cube root at most when we apply the Newton-Raphson Method and we use the first three terms of a Binomial Series and the first four terms of the Taylor's Polynomial. Moreover, we round all of these approximations to the fourth decimal places.

Table 2. Approximating some cube roots for randomly chosen numbers using some methods and the Restriction Method.

$\sqrt[3]{x}$	The calculated value	The approximate value (Newton-Raphson Method)	The absolute value for the error (Newton-Raphson)	The approximate value (Binomial Series)	The absolute value for the error (Binomial Series)	The approximate value (Linear Approximation)	The absolute value for the error (Linear Approximation)	The approximate value (Taylor's Polynomial)	The absolute value for the error (Taylor's Polynomial)	The approximate value (Restriction Method)	The absolute value for the error (Restriction Method)
$\sqrt[3]{2}$	1.2599	1.2599	0.0	1.2840	0.0241	1.3333	0.0734	1.2222	0.0377	1.2963	0.0364
$\sqrt[3]{22}$	2.8020	2.8020	0.0	2.8022	0.0002	2.8148	0.0128	2.8034	0.0014	2.8072	0.0052
$\sqrt[3]{45}$	3.5569	3.5569	0.0	3.5734	0.0165	3.6667	0.1098	3.5185	0.0384	3.5639	0.0070
$\sqrt[3]{60}$	3.9149	3.9149	0.0	3.9149	0.0	3.9167	0.0018	3.9149	0.0	3.9202	0.0053
$\sqrt[3]{97}$	4.5947	4.5947	0.0	4.5953	0.0006	4.6267	0.0320	4.5988	0.0041	4.6083	0.0136
$\sqrt[3]{123}$	4.9732	4.9732	0.0	4.9732	0.0	4.9733	0.0001	4.9732	0.0	4.9740	0.0008
$\sqrt[3]{162}$	5.4514	5.4514	0.0	5.4527	0.0013	5.4933	0.0419	5.4445	0.0067	5.4399	0.0115
$\sqrt[3]{187}$	5.7185	5.7185	0.0	5.7186	0.0001	5.7315	0.0130	5.7195	0.0010	5.7419	0.0234
$\sqrt[3]{721}$	8.9670	8.9670	0.0	8.9670	0.0	8.9671	0.0001	8.9670	0.0	8.9724	0.0054

$\sqrt[3]{941}$	9.7993	9.7993	0.0	9.7993	0.0	9.8033	0.0040	9.7995	0.0002	9.8263	0.0270

From Table (2), we conclude that all of the approximate values obtained by every method are close to the calculated values.

5. Conclusion

In this paper, we present a new method called the Restriction Method for Approximating Cube Roots of rational numbers. The number to find its approximating cube root is restricted between two full cube numbers, then we find the cube roots of them, and by applying on Eq. (1) we find an approximate value to the cube root of the entire number.

The Restriction Method leads mostly to very close values from the calculated values for the cube roots, and it is easy and simple to be applied by students in the preliminary stages; since it does not contain any of complicated assumptions as in other methods like the Newton-Raphson Method, Binomial Series, Linear Approximation and Taylor's Polynomial that any of them needs sufficient knowledge of some concepts, definitions, and techniques like knowing functions, sequences, series, derivatives …etc.

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