Innovative Journal of Mathematics
v. 1 no. 2, 18-29, (2022)
doi:10.55059/ijm.2022.1.2/20

ISSN 2790-3370 (print)
ISSN 2790-3389 (online)

A Companion of Ostrowski inequality for the Stieltjes integral of monotonic functions

Research Article

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Abstract: Some companions of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is assumed to be of r-H-Hölder type on $[a,b]$ and u is of bounded variation on $[a,b]$, are proved. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

MSC: 26D15, 26D20, 41A55

Keywords: Ostrowski's inequality• Bounded variation• Riemann-Stieltjes integral

Received 2022-01-10; Accepted 2022-02-06; Published 2022-04-14

1. Introduction

In [10], Dragomir has proved an Ostrowski inequality for the Riemann-Stieltjes integral, as follows:

Theorem 1.1.
Let $f:[a,b]\to\mathbb{R}$ be a r-H–Hölder type mapping, that is, it satisfies the condition

$$|f(x)-f(y)|\le H\,|x-y|^r,\quad \forall x,y\in[a,b],$$

where, $H>0$ and $r\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\left|f(x)\,(u(b)-u(a))-\int_a^b f(t)\,du(t)\right|\le H\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r\cdot\bigvee_a^b(u) \tag{1}$$

for all $x\in[a,b]$, where, $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac{1}{2}$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r\in(0,1]$.

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In [11], Dragomir has proved the dual case as follows:

Theorem 1.2.
Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then we have the inequality

$$\left|\left(u\left(b\right)-u\left(a\right)\right)f\left(x\right)-\int_a^b f\left(t\right)du\left(t\right)\right| \tag{2}$$
$$\leq H\left[\left(x-a\right)^r\cdot\bigvee_a^x\left(f\right)+\left(b-x\right)^r\cdot\bigvee_x^b\left(f\right)\right]$$
$$\leq H\begin{cases}\left[\left(x-a\right)^r+\left(b-x\right)^r\right]\left[\frac{1}{2}\bigvee_a^b\left(f\right)+\frac{1}{2}\left|\bigvee_a^x\left(f\right)-\bigvee_x^b\left(f\right)\right|\right]\\ \left[\left(x-a\right)^{qr}+\left(b-x\right)^{qr}\right]^{1/q}\left[\left(\bigvee_a^x\left(f\right)\right)^p+\left(\bigvee_x^b\left(f\right)\right)^p\right]^{1/p}\\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r\cdot\bigvee_a^b\left(f\right)\end{cases}$$

In [5], Barnett et al. established some Ostrowski and trapezoid type inequalities for the Stieltjes integral $\int_a^b f\left(t\right)du\left(t\right)$ in the case of Lipschitzian integrators for both Hölder continuous and monotonic integrals are obtained. The dual case is also analyzed. In [6], Cerone et al. proved some Ostrowski type inequalities for the Stieltjes integral where the integrand f is absolutely continuous while the integrator u is of bounded variation. For other results concerning inequalities for Stieltjes integrals, see [3, 7, 8, 14, 16, 18, 20].

Motivated by [17], Dragomir in [13], established the following companion of the Ostrowski inequality for mappings of bounded variation.

Theorem 1.3.
Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:

$$\left|\frac{f\left(x\right)+f\left(a+b-x\right)}{2}-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right|\leq\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b\left(f\right), \tag{3}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $1/4$ is best possible.

For recent results concerning the above companion of Ostrowski's inequality and other related results see [1, 2, 4, 13, 15, 19].

In this paper, we establish a companion of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f\left(t\right)du\left(t\right)$, where f is assumed to be of r-H-Hölder type on $[a,b]$ and u is of bounded variation on $[a,b]$, are given. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

2. The Results

The following companion of Ostrowski's inequality for Riemann-Stieltjes integral holds.

Theorem 2.1.
Let $f:[a,b]\to\mathbb{R}$ be a r-H–Hölder type mapping, where, $H>0$ and $r\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a monotonic nondecreasing function on $[a,b]$. Then we have the inequality

$$\begin{aligned}
&\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \\
&\leq 2H\left\{\left(\frac{a+b}{2}-x\right)^r\left[u\left(\frac{a+b}{2}\right)-u(x)\right]+(x-a)^r\left[u(x)-\frac{u(a)+u(b)}{2}\right]\right\} \\
&\leq 2H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\left[u\left(\frac{a+b}{2}\right)-\frac{u(a)+u(b)}{2}\right],
\end{aligned} \tag{4}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)=f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^{\frac{a+b}{2}}f(t)\,du(t),$$

and

$$\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t)=f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_{\frac{a+b}{2}}^b f(t)\,du(t)$$

Adding the above equalities, we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)+\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t) \\
&\qquad=f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t).
\end{aligned}$$

It is well known that if $p:[c,d]\to\mathbb{R}$ is continuous and $\nu:[c,d]\to\mathbb{R}$ is monotonic nondecreasing, then the Stieltjes integral $\int_c^d p(t)d\nu(t)$ exists and the following inequality holds:

$$\left|\int_c^d p(t)\,d\nu(t)\right|\leq\int_c^d|p(t)|\,d\nu(t). \tag{5}$$

Making use of this property and the fact that f is of r-H-Hölder type on $[a,b]$, we can state that

$$\begin{aligned}
&\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \\
&=\left|\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)+\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t)\right| \\
&\le\left|\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)\right|+\left|\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t)\right| \\
&\le\int_a^{\frac{a+b}{2}}|f(x)-f(t)|\,du(t)+\int_{\frac{a+b}{2}}^b|f(a+b-x)-f(t)|\,du(t) \\
&\le H\int_a^{\frac{a+b}{2}}|x-t|^r\,du(t)+H\int_{\frac{a+b}{2}}^b|a+b-x-t|^r\,du(t). \qquad (6)
\end{aligned}$$

By the integration by parts formula for the Stieltjes integral, we have

$$\begin{aligned}
\int_a^{\frac{a+b}{2}}|x-t|^r\,du(t)&=\int_a^x(x-t)^r\,du(t)+\int_x^{\frac{a+b}{2}}(t-x)^r\,du(t) \\
&=\left(\frac{a+b}{2}-x\right)^r u\left(\frac{a+b}{2}\right)-(x-a)^r u(a) \\
&\quad+r\left[\int_a^x\frac{u(t)}{(x-t)^{1-r}}dt-\int_x^{\frac{a+b}{2}}\frac{u(t)}{(t-x)^{1-r}}dt\right],
\end{aligned}$$

and

$$\begin{aligned}
\int_{\frac{a+b}{2}}^b|a+b-x-t|^r\,du(t)&=\int_{\frac{a+b}{2}}^{a+b-x}(a+b-x-t)^r\,du(t)+\int_{a+b-x}^b(t+x-a-b)^r\,du(t) \\
&=\left(\frac{a+b}{2}-x\right)^r u\left(\frac{a+b}{2}\right)-(x-a)^r u(b) \\
&\quad+r\left[\int_{a+b-x}^b\frac{u(t)}{(t+x-a-b)^{1-r}}dt-\int_{\frac{a+b}{2}}^{a+b-x}\frac{u(t)}{(a+b-x-t)^{1-r}}dt\right].
\end{aligned}$$

Now, by the monotonicity property of u we have

$$\int_a^x\frac{u(t)}{(x-t)^{1-r}}dt\le u(x)\int_a^x\frac{dt}{(x-t)^{1-r}}=\frac{1}{r}(x-a)^r u(x),$$

$$\int_x^{\frac{a+b}{2}}\frac{u(t)}{(t-x)^{1-r}}dt\ge u(x)\int_x^{\frac{a+b}{2}}\frac{dt}{(t-x)^{1-r}}=\frac{1}{r}\left(\frac{a+b}{2}-x\right)^r u(x),$$

$$\int_{a+b-x}^b\frac{u(t)}{(t+x-a-b)^{1-r}}dt\le u(x)\int_{a+b-x}^b\frac{dt}{(t+x-a-b)^{1-r}}=\frac{1}{r}(x-a)^r u(x),$$

and

$$\int_{\frac{a+b}{2}}^{a+b-x} \frac{u(t)}{(a+b-x-t)^{1-r}} dt \geq u(x) \int_{\frac{a+b}{2}}^{a+b-x} \frac{dt}{(a+b-x-t)^{1-r}} = \frac{1}{r}\left(\frac{a+b}{2}-x\right)^r u(x),$$

which follows that

$$\int_a^x \frac{u(t)}{(x-t)^{1-r}} dt - \int_x^{\frac{a+b}{2}} \frac{u(t)}{(t-x)^{1-r}} dt \leq \frac{1}{r}\left[(x-a)^r - \left(\frac{a+b}{2}-x\right)^r\right] u(x)$$

and

$$\begin{aligned}\int_{a+b-x}^{b} \frac{u(t)}{(t+x-a-b)^{1-r}} dt - \int_{\frac{a+b}{2}}^{a+b-x} \frac{u(t)}{(a+b-x-t)^{1-r}} dt \\ \leq \frac{1}{r}\left[(x-a)^r - \left(\frac{a+b}{2}-x\right)^r\right] u(x),\end{aligned}$$

which implies that

$$\begin{aligned}&\left(\frac{a+b}{2}-x\right)^r u\left(\frac{a+b}{2}\right) - (x-a)^r u(a) + r\left[\int_a^x \frac{u(t)}{(x-t)^{1-r}} dt - \int_x^{\frac{a+b}{2}} \frac{u(t)}{(t-x)^{1-r}} dt\right] \\ &\leq \left(\frac{a+b}{2}-x\right)^r u\left(\frac{a+b}{2}\right) - (x-a)^r u(a) + \left[(x-a)^r - \left(\frac{a+b}{2}-x\right)^r\right] u(x) \\ &= \left(\frac{a+b}{2}-x\right)^r \left[u\left(\frac{a+b}{2}\right) - u(x)\right] + (x-a)^r [u(x) - u(a)],\end{aligned}$$

similarly,

$$\begin{aligned}&\left(\frac{a+b}{2}-x\right)^r u\left(\frac{a+b}{2}\right) - (x-a)^r u(b) \\ &\quad + r\left[\int_{a+b-x}^{b} \frac{u(t)}{(t+x-a-b)^{1-r}} dt - \int_{\frac{a+b}{2}}^{a+b-x} \frac{u(t)}{(a+b-x-t)^{1-r}} dt\right] \\ &\leq \left(\frac{a+b}{2}-x\right)^r u\left(\frac{a+b}{2}\right) - (x-a)^r u(b) + \left[(x-a)^r - \left(\frac{a+b}{2}-x\right)^r\right] u(x) \\ &= \left(\frac{a+b}{2}-x\right)^r \left[u\left(\frac{a+b}{2}\right) - u(x)\right] + (x-a)^r [u(x) - u(b)],\end{aligned}$$

which together with (6) proves the first inequality in (4). The second inequality is obvious by the property of max function and we omit the details here. □

The following inequalities are hold:

Corollary 2.1.
Let f and u as in Theorem 2.1. In (4) choose

1. $x = a$, then we get the following trapezoid type inequality

$$\left|f(a)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(b)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \le H\left(\frac{b-a}{2}\right)^r\cdot\bigvee_a^b(u). \quad (7)$$

2. $x = \frac{a+b}{2}$, then we get the following mid-point type inequality

$$\left|(u(b)-u(a))\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \le H\left(\frac{b-a}{2}\right)^r\cdot\bigvee_a^b(u). \quad (8)$$

We may state the following Ostrowski type inequality:

Corollary 2.2.
Let f and u as in Theorem 2.1. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x) = f(x)$, then we have

$$\left|(u(b)-u(a))\,f(x)-\int_a^b f(t)\,dt\right| \le H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b(u), \quad (9)$$

for all $x \in [a, \frac{a+b}{2}]$.

Corollary 2.3.
Let u as in Theorem 2.1, and $f : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, that is,

$$|f(x)-f(y)| \le L\,|x-y|, \quad \forall x,y \in [a,b],$$

where, $L > 0$ is fixed. Then, for all $x \in [a, \frac{a+b}{2}]$, we have the inequality

$$\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \le L\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]\cdot\bigvee_a^b(u). \quad (10)$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Corollary 2.4.
In Theorem 2.1, if u is monotonic on $[a,b]$, and f is of r-H-Hölder type. Then, for all $x \in [a, \frac{a+b}{2}]$, we have the inequality

$$\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \le H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot|u(b)-u(a)|. \quad (11)$$

Corollary 2.5.
Let f be of r-H-Hölder type and $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then we have the inequality

$$\left|f(x)\int_a^{\frac{a+b}{2}} g(s)ds + f(a+b-x)\int_{\frac{a+b}{2}}^b g(s)\,ds - \int_a^b f(t)\,g(t)\,dt\right| \leq H\left[\frac{b-a}{4} + \left|x-\frac{3a+b}{4}\right|\right]^r \|g\|_1, \quad (12)$$

for all $x\in[a,\frac{a+b}{2}]$, where $\|g\|_1=\int_a^b|g(t)|\,dt$.

Proof. Define the mapping $u:[a,b]\to\mathbb{R}$, $u(t)=\int_a^t g(s)ds$. Then u is differentiable on (a,b) and $u'(t)=g(t)$. Using the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)\,g(t)\,dt,$$

and

$$\bigvee_a^b(u) = \int_a^b |u'(t)|\,dt = \int_a^b |g(t)|\,dt$$

□

Remark 2.1.
In Corollary 2.5, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, then we have

$$\left|f(x)\int_a^b g(s)ds - \int_a^b f(t)\,g(t)\,dt\right| \leq H\left[\frac{b-a}{4} + \left|x-\frac{3a+b}{4}\right|\right]^r \|g\|_1, \quad (13)$$

for all $x\in[a,\frac{a+b}{2}]$. For instance, choose $x=\frac{a+b}{2}$, then we get

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b g(s)ds - \int_a^b f(t)\,g(t)\,dt\right| \leq H\left(\frac{b-a}{2}\right)^r \|g\|_1. \quad (14)$$

Theorem 2.2.
Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ be of r-H–Hölder type on $[a,b]$, $r\in(0,1]$. Then we have the inequality

$$\begin{aligned}
&\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right] + f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,du(t)\right| \\
&\leq H\left[(x-a)^r\cdot\bigvee_a^x(f) + \left(\frac{a+b}{2}-x\right)^r\cdot\bigvee_x^{a+b-x}(f) + (x-a)^r\cdot\bigvee_{a+b-x}^b(f)\right] \\
&\leq H\begin{cases}
\left[2(x-a)^r+\left(\frac{a+b}{2}-x\right)^r\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\} \\
\left[2^q(x-a)^{qr}+\left(\frac{a+b}{2}-x\right)^{qr}\right]^{1/q} \\
\times\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^{a+b-x}(f)\right)^p+\left(\bigvee_{a+b-x}^b(f)\right)^p\right]^{1/p}, \; if\ \frac{1}{p}+\frac{1}{q}=1,\ p>1. \\
\left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b(f)
\end{cases} \quad (15)
\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. where, $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

Proof. As u is continuous and f is of bounded variation on $[a, b]$, the following Riemann-Stieltjes integrals exist and, by the integration by parts formula, we can state that

$$\int_a^x (u(t) - u(a))\, df(t) = (u(x) - u(a)) f(x) - \int_a^x f(t)\, du(t),$$

$$\int_x^{a+b-x} \left(u(t) - u\left(\frac{a+b}{2}\right)\right) df(t)$$
$$= \left(u(a+b-x) - u\left(\frac{a+b}{2}\right)\right) f(a+b-x) - \left(u(x) - u\left(\frac{a+b}{2}\right)\right) f(x) - \int_x^{a+b-x} f(t)\, du(t)$$

and

$$\int_{a+b-x}^b (u(t) - u(b))\, df(t)$$
$$= (u(b) - u(a+b-x)) f(a+b-x) - \int_{a+b-x}^b f(t)\, du(t).$$

If we add the above three identities, we obtain

$$\left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(x) + f(a+b-x)\left[u(b) - u\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\, du(t)$$
$$= \int_a^x (u(t) - u(a))\, df(t) + \int_x^{a+b-x} \left(u(t) - u\left(\frac{a+b}{2}\right)\right) df(t) + \int_{a+b-x}^b (u(t) - u(b))\, df(t),$$

for all $x \in [a, \frac{a+b}{2}]$.

Now, using the properties of absolute value, we have:

$$\left|f(x)\left[u\left(\frac{a+b}{2}\right) - u(a)\right] + f(a+b-x)\left[u(b) - u\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\, du(t)\right|$$
$$\leq \left|\int_a^x (u(t) - u(a))\, df(t)\right| + \left|\int_x^{a+b-x} \left(u(t) - u\left(\frac{a+b}{2}\right)\right) df(t)\right|$$
$$+ \left|\int_{a+b-x}^b (u(t) - u(b))\, df(t)\right|$$
$$\leq \int_a^x |u(t) - u(a)|\, df(t) + \int_x^{a+b-x} \left|u(t) - u\left(\frac{a+b}{2}\right)\right| df(t)$$
$$+ \int_{a+b-x}^b |u(t) - u(b)|\, df(t)$$
$$\leq H \int_a^x |t-a|^r\, df(t) + \int_x^{a+b-x} \left|t - \frac{a+b}{2}\right|^r df(t) + \int_{a+b-x}^b |t-b|^r\, df(t)$$
$$\leq H \int_a^x (t-a)^r\, df(t) + \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^r df(t) + \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2}\right)^r df(t)$$
$$+ \int_{a+b-x}^b (b-t)^r\, df(t)$$

and for the last inequality we have used the well-known property if $p : [c, d] \to \mathbb{R}$ is continuous and $\nu : [c, d] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)d\nu(t)$ exists and the following inequality holds:

$$\left|\int_c^d p(t)\, d\nu(t)\right| \leq \sup_{t\in[c,d]} |p(t)| \bigvee_c^d (\nu).$$

As u is of r-H-Hölder type on $[a, b]$, we can state that

$$\sup_{t\in[a,x]} |u(t) - u(a)| \leq \sup_{t\in[a,x]} [H(t-a)^r] = H(x-a)^r,$$

$$\sup_{t\in[x,a+b-x]} \left|u(t) - u\left(\frac{a+b}{2}\right)\right| \leq \sup_{t\in[x,a+b-x]} \left[H\left|t - \frac{a+b}{2}\right|^r\right] = H\left(\frac{a+b}{2} - x\right)^r,$$

and

$$\sup_{t\in[a+b-x,b]} |u(t) - u(b)| \leq \sup_{t\in[a+b-x,b]} [H(b-t)^r] = H(x-a)^r.$$

Now, using (17), we have

$$\begin{aligned}&\left|\left(u\left(\frac{a+b}{2}\right) - u(a)\right) f(x) + \left(u(b) - u\left(\frac{a+b}{2}\right)\right) f(a+b-x) - \int_a^b f(t)\, du(t)\right| \\ &\qquad \leq H\left[(x-a)^r \cdot \bigvee_a^x (f) + \left(\frac{a+b}{2} - x\right)^r \cdot \bigvee_x^{a+b-x} (f) + (x-a)^r \cdot \bigvee_{a+b-x}^b (f)\right] := M(x),\end{aligned}$$

for all $x \in [a, \frac{a+b}{2}]$, and the first inequality in (15) is proved. □

3. An Approximation for the Riemann-Stieltjes Integral

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a, b]$, $h_i = x_{i+1} - x_i$, $(i = 0, 1, 2, \cdots, n-1)$ and $\nu(h) := \max\{h_i | i = 0, 1, 2, \ldots, n-1\}$. Define the general Riemann-Stieltjes sum

$$\begin{aligned}&S(f, u, I_n, \xi) \\ &= \sum_{i=0}^{n-1} f(\xi_i)\left[u\left(\frac{x_i + x_{i+1}}{2}\right) - u(x_i)\right] + f(x_i + x_{i+1} - \xi_i)\left[u(x_{i+1}) - u\left(\frac{x_i + x_{i+1}}{2}\right)\right]\end{aligned} \tag{16}$$

In the following, we establish some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ by its Riemann-Stieltjes sum $S(f, u, I_n, \xi)$.

Theorem 3.1.
Let $u : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then

$$\int_a^b f(t)\,du(t) = S(f,u,I_n,\xi) + R(f,u,I_n,\xi)$$

where, $S(f,u,I_n,\xi)$ is given in (16) and the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$\begin{aligned} |R(f,u,I_n,\xi)| &\le H\left[\frac{1}{4}\nu(h) + \max_{i=0,1,\dots,n-1}\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]^r \cdot \bigvee_a^b (u) \qquad (17)\\ &\le H\left[\frac{1}{2}\nu(h)\right]^r \cdot \bigvee_a^b (u) \end{aligned}$$

Proof. Applying Theorem 2.1 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\begin{aligned} &\Bigg| f(\xi_i)\left[u\left(\frac{x_i + x_{i+1}}{2}\right) - u(x_i)\right] + f(x_i + x_{i+1} - \xi_i)\left[u(x_{i+1}) - u\left(\frac{x_i + x_{i+1}}{2}\right)\right] \\ &\qquad - \int_{x_i}^{x_{i+1}} f(t)\,du(t)\Bigg| \\ &\qquad\qquad \le H\left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]^r \cdot \bigvee_{x_i}^{x_{i+1}} (u), \end{aligned}$$

for all $i \in \{0,1,2,\cdots,n-1\}$.

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned} &|R(f,u,I_n,\xi)| \\ &= \sum_{i=0}^{n-1}\Bigg| f(\xi_i)\left[u\left(\frac{x_i + x_{i+1}}{2}\right) - u(x_i)\right] + f(x_i + x_{i+1} - \xi_i)\left[u(x_{i+1}) - u\left(\frac{x_i + x_{i+1}}{2}\right)\right] \\ &\quad - \int_{x_i}^{x_{i+1}} f(t)\,du(t)\Bigg| \\ &\le H\sum_{i=0}^{n-1}\left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]^r \cdot \bigvee_{x_i}^{x_{i+1}} (u) \\ &\le H \sup_{i=0,1,\dots,n-1}\left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]^r \cdot \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}} (u). \end{aligned}$$

However,

$$\sup_{i=0,1,\dots,n-1}\left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]^r \le \left[\frac{1}{4}\nu(h) + \sup\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]^r,$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}} (u) = \bigvee_a^b (u).$$

which completely proves the first inequality in (17).

For the second inequality, we observe that

$$\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right| \le \frac{1}{4}h_i$$

for all $i \in \{0, 1, 2, \cdots, n-1\}$. which completes the proof. □

Corollary 3.1.
In Theorem 3.1, additionally, if f is symmetric about the x-axis, then we have $S(f, u, I_n, \xi)$ reduced to be

$$S(f, u, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i)\,[u(x_{i+1}) - u(x_i)]. \tag{18}$$

Then

$$\int_a^b f(t)\,du(t) = S(f, u, I_n, \xi) + R(f, u, I_n, \xi)$$

where, $S(f, u, I_n, \xi)$ is given in (18) and the remainder $R(f, u, I_n, \xi)$ satisfies the bound in (17).

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