

# A Short note on the fractional trapezium type integral inequalities

Research Article

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**Abstract:** The authors have examined a large number of mathematical articles that deal with the expansion of the convexity technique and its various strategies, and focusing on it, they have determine the connection that can be developed among fractional trapezium type inequalities for generalized convex function. The authors advance this direction by examining the Hermite-Hadamard inequality by using  $h$ -preinvex functions.

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## 1. Introduction

The theory of convexity has evolved even more in recent years as a result of its broad application in various fields of science, as illustrated in the following references ((Grinalatt, & Linnainmaa, 2011) [5], Nicolescu & peerson, 2006) [11], (PecariĂ'c, Proschan & Tong, 1992) [14], (Ruel & Ayres, 1999) [15]. Generalized convexity and its application like  $h$ -convexity,  $\eta$ -convexity, (s,m)-convexity and others, are discuss in the following references(Hernandez Hernandez & Vivas-Cortez, 2019) [6], (Liu, Wen & Park, 2016) [10], (Noor, 2006) [12], (Vivas-Cortez, 2016) [17], ( Vivas, Hernandez Hernandez & Merentes, 2016) [16]. Let  $f : p \subseteq R \rightarrow R$  is a convex function and  $u, v \in I$  with  $u < v$ .

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{d-c} \int_c^d f(s)ds \leq \frac{f(u)+f(v)}{2} \quad (1)$$

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The above inequality, Hermite-Hadamard inequality is one of the most important inequality for convex function. This inequality has large applicability in the domain of stats and probability (Pecari’c, Proshan & Tong, 1992) [14] also include domain of functional analysis (Nicolescu & Peerson, 2006) [11]. In current years various analyst have investigated under the field of modern version in the advancement of the understanding of convex function. As represented in the following references (Aslani, Delavar, & Vaezpour, 2018) [1], (Chen & Wu , 2016) [2],(Delavar & De La Sen , 2016) [4], (kashuri & Liko, 2019) [9], (Omotoyinbo& Mogbademu, 2014) [13], (Xi & Qi, 2012) [18]). Further the extent progress of the method of convex function has been linked in the area of integral inequalities represent in the following paper(Vivas, Hernandez Hernandez & Merentes, 2016) [16], (Vivas-Cortez, 2016) [17]. Influenced by the valuable work stated above, we modified the following work by examining the Hermite-Hadamard inequality using h-preinvex function.

## 2. Preliminaries

We identify the following definition linked with the h pre-invex .

### Definition 2.1.

If the given inequality true for the set  $k \subseteq \mathbb{R}^n$  and  $u, v \in \mathbb{K}$  then a function f is known as h pre-invex

$$f(u + s\eta(v, u)) \leq h(1 - s)f(u) + h(s)f(v)$$

where  $\eta(u, v) : k * k \rightarrow \mathbb{R}$  and  $s \in (0, 1)$  and  $h \neq 0$  be a non negative function  $h : [0, 1] \rightarrow \mathbb{R}$  .

### Remark 2.1.

We obtain definition of classical convex function by putting  $h(s)=s$  and  $\eta(v, u) = v - u$  in definition 1.

### Definition 2.2.

(Kermausor & Nwaeze, 2020) [8] The fractional integrals of order  $\alpha > 0$  of f on the left- and right- sides of Riemann-Liouville are denoted as

$${}_k J_{c^+}^\alpha f(u) := \frac{1}{k\Gamma_k(\alpha)} \int_c^u (u - s)^{\frac{\alpha}{k}-1} f(s) ds, u > c$$

and

$${}_k J_{d^-}^\alpha f(u) := \frac{1}{k\Gamma_k(\alpha)} \int_u^d (s - u)^{\frac{\alpha}{k}-1} f(s) ds, u < d$$

$\Gamma_k$  is the k-gamma function presented by

$$\Gamma_k(u) := \int_0^\infty s^{u-1} e^{-\frac{s^k}{k}} ds, Re(u) > 0$$

where  $k > 0$ , with the properties that  $\Gamma_k(u + k) = u\Gamma_k(u)$ ,  $\Gamma_k(k) = 1$  and  $\Gamma_1(u) = \Gamma(u)$ .

### Definition 2.3.

Let  $[c, d] \subset \mathbb{R}$  be a finite interval. The katugampola fractional integrals of order  $\alpha > 0$  of  $f \in X_c^\rho(a, b)$  on the left- and right-sides are therefore denoted by

$${}^\rho I_{c^+}^\alpha f(u) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_c^u \frac{s^{\rho-1}}{(u^\rho - s^\rho)^{1-\alpha}} f(s) ds$$

$${}^\rho I_{d-}^\alpha f(u) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_u^d \frac{s^{\rho-1}}{(s^\rho - u^\rho)^{1-\alpha}} f(s) ds$$

with  $c < u < d, \rho > 0$ , if the integrals exist.

**Remark 2.2.**

Let  $\alpha > 0$  and  $\rho > 0$ . Then for  $u > c$

$$\lim_{\rho \rightarrow 1} {}^\rho I_{c+}^\alpha f(u) = J_{c+}^\alpha f(u)$$

The same is true for right-handed operators.

**Definition 2.4.**

Let  $k \subseteq \mathbb{R}^n$  be the set and  $u, v \in \mathbb{K}$  then a function  $f$  is said to be  $(\psi, h)$  pre-invex if

$$f(u + \xi e^{\psi}(\eta(v, u))) \leq h(1 - \xi)f(u) + h(\xi)f(v)$$

where  $\eta(u, v) : k * k \rightarrow \mathbb{R}$  and  $s \in (0, 1)$  and  $h \neq 0$  be a non negative function  $h : [0, 1] \rightarrow \mathbb{R}$ .

**Remark 2.3.**

If we take  $h(\xi) = \xi$  and  $\eta(v, u) = v - u$  in Definition 4 then we get  $\psi$  convex function.

**Theorem 2.1.**

Let  $\alpha > 0$  and let  $f : [c, d] \rightarrow \mathbb{R}$  be a positive function with  $0 \leq c < d$  and  $f \in L[c, d]$ . If  $f$  is a convex function on  $[c, d]$ , then the given inequality true:

$$f\left(\frac{c+d}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2[d-c]^\alpha} [J_{a+}^\alpha f(d) + J_{d-}^\alpha f(c)] \leq \frac{f(c) + f(d)}{2}$$

### 3. Main Results

In this section, we generalize the results of (Jleli, O'Regan & Samet, 2016) [7]. Let  $f : [c, c + \eta(d, c)] \rightarrow R$  be a given function, where  $0 < c < c + \eta(d, c) < \infty$ . We define  $F(x) = f(x) + f(2c + \eta(d, c) - x)$ . Then it is easy to show that if  $f(x)$  is convex on  $[c, c + \eta(d, c)]$ ,  $F(x)$  is also convex. The function  $F$  has several interesting properties, especially,

1.  $F(x)$  is symmetric to  $(2c + \eta(d, c))/2$
2.  $F(c) = F(c + \eta(d, c)) = f(c) + f(c + \eta(d, c))$
3.  $F\left(\frac{2c + \eta(d, c)}{2}\right) = 2f\left(\frac{2c + \eta(d, c)}{2}\right)$

**Theorem 3.1.**

If  $f$  is a convex function on  $[c, c + \eta(d, c)]$  and  $f \in L[c, c + \eta(d, c)]$ . Then  $F(x)$  is also integrable, and the following inequalities hold

$$F\left(\frac{2c + \eta(d, c)}{2}\right) \leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2[\eta(d, c)^\rho]^\alpha} [{}^\rho I_{c+}^\alpha F(c + \eta(d, c)) + {}^\rho I_{[c + \eta(d, c)]-}^\alpha F(c)] \leq \frac{F(c) + F(c + \eta(d, c))}{2} \tag{2}$$

with  $\alpha > 0$  and  $\rho > 0$ .

*Proof.* Since  $f(x)$  is a convex function on  $[c,d]$ , we have for  $x, y \in [c, d]$

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

Set  $x = c + s\eta(d, c)$  and  $y = c + (1 - s)\eta(d, c)$  then

$$2f\left(\frac{2c + \eta(d, c)}{2}\right) \leq f(c + s\eta(d, c)) + f(c + (1 - s)\eta(d, c))$$

Using the notation of  $F(x)$ , we have

$$F\left(\frac{2c + \eta(d, c)}{2}\right) \leq F(c + (1 - s)\eta(d, c)) \tag{3}$$

Multiplying both side of (3.3) by

$$\frac{(c + (1 - s)\eta(d, c))^{\rho-1}}{[(c + \eta(d, c))^\rho - (c + (1 - s)\eta(d, c))^\rho]^{1-\alpha}} \tag{4}$$

integrating the resulting inequality with respect to  $s$  over  $[0,1]$ , we get

$$\begin{aligned} F\left(\frac{2c + \eta(d, c)}{2}\right) \frac{[\eta(d, c)^\rho]^\alpha}{\alpha\rho\eta(d, c)} &\leq \int_0^1 \frac{(c + (1 - s)\eta(d, c))^{\rho-1}}{[(c + \eta(d, c))^\rho - (c + (1 - s)\eta(d, c))^\rho]^{1-\alpha}} F(c + (1 - s)\eta(d, c)) ds \\ &= \frac{1}{\eta(d, c)} \int_c^{c+\eta(d, c)} \frac{u^{\rho-1}}{[(c + \eta(d, c))^\rho - u^\rho]^{1-\alpha}} F(u) du \\ &= \frac{\Gamma(\alpha)\rho^{\alpha-1}}{\eta(d, c)} [{}^\rho I_{c+}^\alpha F(c + \eta(d, c))] \\ F\left(\frac{2c + \eta(d, c)}{2}\right) &\leq \frac{\Gamma(\alpha + 1)\rho^\alpha}{[\eta(d, c)^\rho]^\alpha} [{}^\rho I_{c+}^\alpha F(c + \eta(d, c))] \end{aligned} \tag{5}$$

Similarly multiplying both sides of (3.3) by

$$\frac{[c + (1 - s)\eta(d, c)]^{\rho-1}}{[(c + (1 - s)\eta(d, c))^\rho - c^\rho]^{1-\alpha}} \tag{6}$$

integrating the resulting inequality over  $[0,1]$ , we get

$$F\left(\frac{2c + \eta(d, c)}{2}\right) \leq \frac{\Gamma(\alpha + 1)\rho^\alpha}{[\eta(d, c)^\rho]^\alpha} [{}^\rho I_{(c+\eta(d, c))^-}^\alpha F(c)] \tag{7}$$

By adding (3.5) and (3.7), we obtain

$$F\left(\frac{2c + \eta(d, c)}{2}\right) \leq \frac{\Gamma(\alpha + 1)\rho^\alpha}{2[\eta(d, c)^\rho]^\alpha} [{}^\rho I_{c+}^\alpha F(c + \eta(d, c)) + {}^\rho I_{(c+\eta(d, c))^-}^\alpha F(c)] \tag{8}$$

The first inequality of (3.2) is proved.

For the second part, since  $f$  is convex function, then for  $t \in [0, 1]$ , we have

$$f(c + s\eta(d, c)) + f(c + (1 - s)\eta(d, c)) \leq f(c) + f(c + \eta(d, c))$$

Using the notation of  $F(x)$ , we then have

$$F(c + (1 - s)\eta(d, c)) \leq \frac{F(c) + F(c + \eta(d, c))}{2} \tag{9}$$

Multiplying both sides of (3.9) by factor (3.4) and integrating the resulting inequality over  $[0, 1]$ , with respect to  $s$ , we get

$$\begin{aligned} \frac{\Gamma(\alpha)\rho^{\alpha-1}}{\eta(d, c)} [{}^\rho I_{c+}^\alpha F(c + \eta(d, c))] &\leq \frac{[\eta(d, c)^\rho]^\alpha}{\alpha\rho(\eta(d, c))} \frac{F(c) + F(c + \eta(d, c))}{2} \\ \frac{\Gamma(\alpha + 1)\rho^\alpha}{[\eta(d, c)^\rho]^\alpha} [{}^\rho I_{c+}^\alpha F(c + \eta(d, c))] &\leq \frac{F(c) + F(c + \eta(d, c))}{2} \end{aligned} \tag{10}$$

Similarly multiplying both sides of (3.9) by factor (3.6) and integrating the resulting inequality over  $[0, 1]$ , with respect to  $s$ , we get

$$\frac{\Gamma(\alpha + 1)\rho^\alpha}{[\eta(d, c)^\rho]^\alpha} [{}^\rho I_{(c+\eta(d, c))^-}^\alpha F(c)] \leq \frac{F(c) + F(c + \eta(d, c))}{2} \tag{11}$$

Adding (3.10) and (3.11), we get

$$\frac{\Gamma(\alpha + 1)\rho^\alpha}{2[\eta(d, c)^\rho]^\alpha} [{}^\rho I_{c+}^\alpha F(c + \eta(d, c)) + {}^\rho I_{(c+\eta(d, c))^-}^\alpha F(c)] \leq \frac{F(c) + F(c + \eta(d, c))}{2}$$

The proof is complete. □

**Remark 3.1.**

Theorem 2 is a generalization of Hermite-Hadamard inequality. Putting  $\rho \rightarrow 1$  in (3.2)

1.  ${}^\rho I_{c+}^\alpha F(c + \eta(d, c)) = \frac{1}{\Gamma(\alpha)} \int_c^{c+\eta(d, c)} [(c + \eta(d, c)) - u]^{\alpha-1} F(u) du = J_{c+}^\alpha f(c + \eta(d, c)) + J_{(c+\eta(d, c))^-}^\alpha f(c)$
2.  ${}^\rho I_{(c+\eta(d, c))^-}^\alpha F(c) = \frac{1}{\Gamma(\alpha)} \int_c^{c+\eta(d, c)} [u - c]^{\alpha-1} F(u) du = J_{(c+\eta(d, c))^-}^\alpha f(c) + J_{c+}^\alpha f(c + \eta(d, c))$

We get the Riemann-Liouville form of Hermite-Hadamard inequality of Theorem 1.

In order to prove Theorem 3, we need the following lemma.

**Lemma 3.1.**

Let  $f : [c, c + \eta(d, c)] \rightarrow R$  be a differentiable mapping on  $(c, c + \eta(d, c))$  with  $c < c + \eta(d, c)$ . If  $f' \in L[c, c + \eta(d, c)]$ , then  $F$  is also differentiable and  $F' \in L[c, c + \eta(d, c)]$ , and the following equality holds:

$$\begin{aligned} & \frac{F(c) + F(c + \eta(d, c))}{2} - \frac{\Gamma(\alpha + 1)\rho^\alpha}{2[\eta(d, c)\rho]^\alpha} [\rho I_{c+}^\alpha F(c + \eta(d, c)) + \rho I_{(c+\eta(d, c))-}^\alpha F(c)] = \\ & \frac{\eta(d, c)}{2[\eta(d, c)\rho]^\alpha} \int_0^1 K(s)F'(c + (1 - s)\eta(d, c))ds \end{aligned} \tag{12}$$

with  $\alpha > 0$  and  $\rho > 0$ . Where  $K(s) = [(c + (1 - s)\eta(d, c))^\rho - c^\rho]^\alpha - [(c + \eta(d, c))^\rho - (c + (1 - s)\eta(d, c))^\rho]^\alpha$ .

*Proof.*

$$\begin{aligned} I &= \int_0^1 K(s)F'(c + (1 - s)\eta(d, c))ds \\ &= \int_0^1 [(c + (1 - s)\eta(d, c))^\rho - c^\rho]^\alpha F'(c + (1 - s)\eta(d, c))ds - \\ & \int_0^1 [(c + \eta(d, c))^\rho - (c + (1 - s)\eta(d, c))^\rho]^\alpha F'(c + (1 - s)\eta(d, c))ds \\ &= I_1 + I_2 \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 [(c + (1 - s)\eta(d, c))^\rho - c^\rho]^\alpha F'(c + (1 - s)\eta(d, c))ds \\ &= \frac{1}{\eta(d, c)} \int_c^{c+\eta(d, c)} [u^\rho - c^\rho]^\alpha dF(u) \\ &= \left[ \frac{(u^\rho - c^\rho)^\alpha F(u)}{\eta(d, c)} \right]_c^{c+\eta(d, c)} - \frac{\alpha\rho}{\eta(d, c)} \int_c^{c+\eta(d, c)} \frac{u^{\rho-1}}{(u^\rho - c^\rho)^{1-\alpha}} F(u)du \\ &= \frac{(\eta(d, c)\rho)^\alpha}{\eta(d, c)} F(c + \eta(d, c)) - \frac{\Gamma(\alpha + 1)\rho^\alpha}{\eta(d, c)} \rho I_{(c+\eta(d, c))-}^\alpha F(c) \end{aligned} \tag{13}$$

Similarly

$$\begin{aligned} I_2 &= - \int_0^1 [(c + \eta(d, c))^\rho - (c + (1 - s)\eta(d, c))^\rho]^\alpha F'(c + (1 - s)\eta(d, c))ds \\ &= \frac{(\eta(d, c)\rho)^\alpha}{\eta(d, c)} F(c) - \frac{\Gamma(\alpha + 1)\rho^\alpha}{\eta(d, c)} \rho I_{c+}^\alpha F(c + \eta(d, c)) \end{aligned} \tag{14}$$

Adding (3.13) and (3.14), we get

$$I = \frac{(\eta(d, c)\rho)^\alpha}{\eta(d, c)} [F(c) + F(c + \eta(d, c))] - \frac{\Gamma(\alpha + 1)\rho^\alpha}{\eta(d, c)} [\rho I_{c+}^\alpha F(c + \eta(d, c))] + \rho I_{(c+\eta(d, c))-}^\alpha F(c)$$

Then, multiplying both sides by  $\frac{\eta(d, c)}{2(\eta(d, c)\rho)^\alpha}$  we obtain equality (3.12). □

We are now ready to prove the following Hermite-Hadamard type inequality.

**Theorem 3.2.**

Let  $f : [c, c + \eta(d, c)] \rightarrow R$  be a differentiable mapping on  $(c, c + \eta(d, c))$  with  $a < b$  and  $f' \in L[c, c + \eta(d, c)]$ . Then  $F$  is also differentiable and  $F' \in L[c, c + \eta(d, c)]$ . If  $|f'|$  is convex on  $[c, c + \eta(d, c)]$ , then the following inequality holds:

$$\begin{aligned} & \frac{F(c) + F(c + \eta(d, c))}{2} - \frac{\Gamma(\alpha + 1)\rho^\alpha}{2[\eta(d, c)\rho]^\alpha} [\rho I_{c+}^\alpha F(c + \eta(d, c)) + \rho I_{(c+\eta(d,c))^-}^\alpha F(c)] = \\ & \frac{\eta(d, c)}{2[\eta(d, c)\rho]^\alpha} \int_0^1 |K(s)ds|(2|f'(c)| + |f'(\eta(d, c))|) \end{aligned} \tag{15}$$

with  $\alpha > 0$  and  $\rho > 0$ . Where  $K(s) = [(c + (1 - s)\eta(d, c))^\rho - c^\rho]^\alpha - [(c + \eta(d, c))^\rho - (c + (1 - s)\eta(d, c))^\rho]^\alpha$ .

*Proof.* Notice that  $F'(x) = f'(x) - f'(c + \eta(d, c) - x)$ . By the convexity of  $|f'|$ , we have

$$\begin{aligned} F'(c + (1 - s)\eta(d, c)) &= f'((c + (1 - s)\eta(d, c)) - f'(c + s\eta(d, c))) \\ &\leq |f'(c)| + (1 - s)|f'(\eta(d, c))| + |f'(c)| + s|f'(\eta(d, c))| \\ &= 2|f'(c)| + |f'(\eta(d, c))| \end{aligned} \tag{16}$$

By inequalities (3.12) and (3.16) we get

$$\begin{aligned} & \left| \frac{F(c) + F(c + \eta(d, c))}{2} - \frac{\Gamma(\alpha + 1)\rho^\alpha}{2[\eta(d, c)\rho]^\alpha} [\rho I_{c+}^\alpha F(c + \eta(d, c)) + \rho I_{(c+\eta(d,c))^-}^\alpha F(c)] \right| \leq \\ & \frac{\eta(d, c)}{2[\eta(d, c)\rho]^\alpha} \int_0^1 |K(s)||F'(c + (1 - s)\eta(d, c))|ds \\ & \leq \frac{\eta(d, c)}{2[\eta(d, c)\rho]^\alpha} \int_0^1 |K(s)|ds(2|f'(c)| + |f'(\eta(d, c))|) \end{aligned}$$

This complete the proof. □

## 4. Concluding Remarks

Here we determine major results related to Hermite-Hadamard inequality using  $h$  pre-invexity. A couple of results in the previous research paper are special cases of few of our results. The modified integral inequalities give a more precise approximation than some of the preceding ones.

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