Innovative Journal of Mathematics
v. 1 no. 2, 30-43, (2022)
doi:10.55059/ijm.2022.1.2/23

ISSN 2790-3370 (print)
ISSN 2790-3389 (online)

# Some new generalized Ostrowski type inequalities with new error bounds

**Research Article**

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**Abstract:** In this paper, we generalize Ostrowski type inequalities for twice differentiable mappings. Some previous results can be recaptured as a special cases of the inequalities obtained here. Furthermore, perturbed mid-point inequality and perturbed trapezoid inequality are also obtained. Applications in numerical integrations and some special means are also discussed.

**MSC:** 65D30, 32A26

**Keywords:** Ostrowski inequality • Kernel and numerical integration

Received 2022-02-03; Accepted 2022-03-31; Published 2022-04-14

## 1. Introduction

Inequalities have been proved to be an applicable tool for the development of many branches of Mathematics. From the past few decades, its importance has been increased noticeably and it is now treated as an independent branch of Mathematics. Uptill now a vast number of research papers and books have been devoted to inequalities. A. M. Ostrowski (1893-1986) in 1938, gave a helpful and vital integral inequality known as Ostrowski's inequality [22].

M. W. Alomari [1]-[4] worked on generlizations of Ostrowski's type inequalities. S. I. Butt [6] gave the Jensen-Grüss inequality and its applications. S. S. Dragomir *et.al* [9] and [10] presented inequality of Ostrowski type for $||\cdot||_1$ and applications of Ostrowski's inequality to numerical quadrature rules and special means. A. Qayyum *et.al*

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[25] gave new inequalities of Ostrowski's type. P. Cerone *et.al* [7] and [8] pointed out an inequality of Ostrowski's type for $L_\infty(a,b)$, $L_1(a,b)$ and $L_p(a,b)$. A. Qayyum *et.al* [23] and [24] offered Ostrowski' s type inequalities which were the generalization of the inequalities given in [5]. Different researchers [11]-[21] worked on refinement of Ostrowski's type inequalities and its applications. From the above work, we develop new gernalized inequalities for different norms e.g $\|g''\|_\infty$, $\|g''\|_1$ and $\|g''\|_p$. In the end, we give applications for some special means and in numerical integration.

## 2. Results and Discussion

**Theorem 2.1.**
*Let $g:[a,\dot{c}]\to R$ be continuous on $[a,\dot{c}]$ and twice differentiable mapping on $(a,\dot{c})$. Then*

$$\Bigg|\left(1-\frac{2h}{ꝁ}\right)\left[g(x)-\left(x-\frac{a+\dot{c}}{2}\right)g'(x)\right] + \frac{h}{ꝁ}\left[(g(a)+g(\dot{c}))-\frac{h(\dot{c}-a)}{2ꝁ}\left(g'(\dot{c})-g'(a)\right)\right]-\frac{1}{\dot{c}-a}\int_a^{\dot{c}} g(t)dt\Bigg|$$
$$\leq \begin{cases} \left[3\left(1-\frac{2h}{ꝁ}\right)^2+1\right]\frac{(\dot{c}-a)^2}{24}\|g''\|_\infty & \text{if } g''\in L_\infty(a,\dot{c}) \\ \frac{\dot{c}-a}{2}\left(1-\frac{2h}{ꝁ}\right)^2\|g''\|_1 & \text{if } g''\in L_1(a,\dot{c}) \\ \frac{(\dot{c}-a)^{1+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}}\left[\left(1-\frac{2h}{ꝁ}\right)^{2q+1}+2\left(\frac{h}{ꝁ}\right)^{2q+1}\right]^{\frac{1}{q}}\|g''\|_p & \text{if } g''\in L_p(a,\dot{c}) \end{cases} \tag{1}$$

*holds for all $x\in\left[a+h\frac{\dot{c}-a}{ꝁ},\dot{c}-h\frac{\dot{c}-a}{ꝁ}\right]$, where $\frac{1}{p}+\frac{1}{q}=1$ and $h\in[0,\ 1]$, $ꝁ=1,2,3,...,n$.*

*Proof.* Define $K(\cdot\,,\cdot):[a,\dot{c}]^2\to R$ such that

$$K(x,t)=\begin{cases}\alpha\left[t-\left(a+h.\frac{\dot{c}-a}{ꝁ}\right)\right]^2, & \text{if } t\in[a,x] \\ \alpha\left[t-\left(\dot{c}-h.\frac{\dot{c}-a}{ꝁ}\right)\right]^2, & \text{if } t\in(x,\dot{c}]\end{cases} \tag{2}$$

By using (2) and after some calculations, we get the following identity:

$$\int_a^{\dot{c}} g(t)dt = (\dot{c}-a)(1-\frac{2h}{ꝁ})\left[g(x)-\left(x-\frac{a+\dot{c}}{2}\right)g'(x)\right] + \frac{h(\dot{c}-a)}{ꝁ}\left[(g(a)+g(\dot{c}))-\frac{h(\dot{c}-a)}{2ꝁ}\left(g'(\dot{c})-g'(a)\right)\right] + \frac{1}{2\alpha}\int_a^{\dot{c}} K(x,t)g''(t)dt. \tag{3}$$

We can write (3) as

$$\left|\left(1-\frac{2h}{ķ}\right)\left[g(x)-\left(x-\frac{a+\dot{c}}{2}\right)g'(x)\right]\right.$$
$$\left.+\frac{h}{ķ}\left[\left(g(a)+g\left(\dot{c}\right)\right)\ -\frac{h(\dot{c}-a)}{2ķ}\left(g'(\dot{c})-g'(a)\right)\right]-\frac{1}{\dot{c}-a}\int\limits_a^{\dot{c}}g(t)dt\right|$$
$$\leq\frac{1}{2\alpha(\dot{c}-a)}\left(\int\limits_a^{\dot{c}}\left|K\left(x,t\right)\right|dt\right)\left|\left|\ g''\right|\right|_{\infty} \tag{4}$$

where

$$\int\limits_a^{\dot{c}}\left|K\left(x,t\right)\right|dt$$
$$=2\alpha\left(\dot{c}-a\right)\left[\left(1-\frac{2h}{ķ}\right)\left\{\frac{\left(1-\frac{2h}{ķ}\right)^2\left(\dot{c}-a\right)^2}{24}\right.\right.$$
$$\left.\left.+\frac{1}{2}\left(x-\frac{a+\dot{c}}{2}\right)^2\right\}+\frac{h^3\left(\dot{c}-a\right)^2}{3ķ^3}\right]. \tag{5}$$

Using (5) in (4), we can find the first inequality in (1).

By using (2) and (3), we get

$$\left|\left(1-\frac{2h}{ķ}\right)\left[g(x)-\left(x-\frac{a+\dot{c}}{2}\right)g'(x)\right]\right.$$
$$\left.+\frac{h}{ķ}\left[\left(g(a)+g\left(\dot{c}\right)\right)\ -\frac{h(\dot{c}-a)}{2ķ}\left(g'(\dot{c})-g'(a)\right)\right]-\frac{1}{\dot{c}-a}\int\limits_a^{\dot{c}}g(t)dt\right|$$
$$\leq\frac{1}{2(\dot{c}-a)}max\left\{\left[x-\left(a+h.\frac{\dot{c}-a}{ķ}\right)\right]^2,\right.$$
$$\left.\left[\left(\dot{c}-h.\frac{\dot{c}-a}{ķ}\right)-x\right]^2\right\}\left|\left|\ g''\right|\right|_1.$$

After simplification, we get second inequality in (1).

Again using (2) and (3), we get

$$\left|\left(1-\frac{2h}{ķ}\right)\left[g(x)-\left(x-\frac{a+\dot{c}}{2}\right)g'(x)\right]\right.$$
$$\left.+\frac{h}{ķ}\left[\left(g(a)+g\left(\dot{c}\right)\right)\ -\frac{h(\dot{c}-a)}{2ķ}\left(g'(\dot{c})-g'(a)\right)\right]-\frac{1}{\dot{c}-a}\int\limits_a^{\dot{c}}g(t)dt\right|$$
$$\leq\frac{1}{2\alpha(\dot{c}-a)}\left(\int\limits_a^{\dot{c}}K^q\left(x,t\right)dt\right)^{\frac{1}{q}}\left|\left|\ g''\right|\right|_p$$

where

$$\int_{a}^{\dot{c}} K^{q}(x,t)\,dt \leq \frac{\alpha^{q}}{2q+1}(\dot{c}-a)^{2q+1}\left[\left(1-\frac{2h}{ķ}\right)^{2q+1}+2\left(\frac{h}{ķ}\right)^{2q+1}\right].$$

After simplification, we get third inequality in (1).

Hence proved our main result (1). □

**Remark 2.1.**
For $h=0$ in (1), we obtain the result obtained by Barnett *et.al* [5], P. Cerone *et.al* in [7] and [8], and for ķ$=2$ in (1), we get the result obtained by A. Qayyum *et.al* in [23] and [24] which indicates special cases. Hence for different values of $h$ and ķ, we can get variety of results.

## 3. Application for Some Special Means

**Remark 3.1.**
Consider $g:(0,\infty)\to R$ such that

$$g(x)=x^{r},\ r\in R\backslash\{-1,0\}$$

then

$$\frac{1}{\dot{c}-a}\int_{a}^{\dot{c}} g(x)\,dx = L_{r}^{r}(a,\dot{c}),$$

$$g(a)+g(\dot{c}) = 2A\left(a^{r},\dot{c}^{r}\right),$$

$$g'(\dot{c})-g'(a) = r(r-1)(\dot{c}-a)\,L_{r-2}^{r-2}(a,\dot{c})$$

$$\|g''\|_{\infty} = |r(r-1)|\,\delta_{r}(a,\dot{c})$$

where

$$\delta_{r}(a,\dot{c}) = \begin{cases} \dot{c}^{r-2} & if \quad r\in(2,\infty) \\ a^{r-2} & if \quad r\in(-\infty,2)\backslash\{-1,0\} \end{cases}$$

So, (1) gives

$$\left|\left(1-\frac{2h}{ķ}\right)\left[x^{r}-r(x-A)x^{r-1}\right] + \frac{2h}{ķ}\left[A(a^{r},\dot{c}^{r})-\frac{hr(r-1)(\dot{c}-a)^{2}}{4ķ}L_{r-2}^{r-2}\right]-L_{r}^{r}\right| \leq \frac{|r(r-1)|(\dot{c}-a)^{2}}{24}\left[3\left(1-\frac{2h}{ķ}\right)^{2}+1\right]\delta_{r}(a,\dot{c}) \tag{6}$$

Choosing $x = A(a, \dot{c})$ in (6), we get

$$\left| \left(1 - \frac{2h}{ķ}\right) A^r + \frac{2h}{ķ} \left[ A(a^r, \dot{c}^r) - \frac{hr(r-1)(\dot{c}-a)^2}{4ķ} L_{r-2}^{r-2} \right] - L_r^r \right| \leq \frac{|r(r-1)|(\dot{c}-a)^2}{24} \left[ 3\left(\frac{2h}{ķ} - \frac{1}{2}\right)^2 + \frac{1}{4} \right] \delta_r (a, \dot{c})$$

**Remark 3.2.**
Consider $g : (0, \infty) \to R$ such that

$$g(x) = \frac{1}{x}, \ x \in \left[ a + h.\frac{\dot{c}-a}{ķ}, \dot{c} - h.\frac{\dot{c}-a}{ķ} \right] \subset (0, \infty)$$

then

$$\frac{1}{\dot{c}-a} \int_a^{\dot{c}} g(x)\, dx = L^{-1}(a, \dot{c})$$

$$g(a) + g(\dot{c}) = \frac{2}{H(a, \dot{c})} = \frac{2A(a, \dot{c})}{G^2(a, \dot{c})}$$

$$g'(\dot{c}) - g'(a) = \frac{2(\dot{c}-a)}{H(a, \dot{c})\, G^2(a, \dot{c})} = \frac{2(\dot{c}-a)\, A(a, \dot{c})}{G^4(a, \dot{c})}$$

$$||g''||_\infty = \frac{2}{a^3}$$

So, (1) gives

$$\left| \left(1 - \frac{2h}{ķ}\right) \left[2 - \frac{A}{x}\right] \frac{1}{x} + \frac{2hA}{ķG^2} \left[1 - \frac{h(\dot{c}-a)^2}{2ķG^2}\right] - L^{-1} \right| \leq \frac{(\dot{c}-a)^2}{12a^3} \left[ 3\left(1 - \frac{2h}{ķ}\right)^2 + 1 \right] \tag{7}$$

Choosing $x = A(a, \dot{c})$ in (7), we get

$$\left| \left(1 - \frac{2h}{ķ}\right) \frac{1}{A} + \frac{2hA}{ķG^2} \left[1 - \frac{h(\dot{c}-a)^2}{2ķG^2}\right] - L^{-1} \right| \leq \frac{(\dot{c}-a)^2}{12a^3} \left[ 3\left(\frac{2h}{ķ} - \frac{1}{2}\right)^2 + \frac{1}{4} \right].$$

Choosing $x = L(a, \dot{c})$ in (7), we get

$$\left| \left[ \left(1 - \frac{2h}{ķ}\right) \left(2 - \frac{A}{L}\right) - 1 \right] \frac{1}{L} + \frac{2h}{ķH} \left[1 - \frac{h(\dot{c}-a)^2}{2ķG^2}\right] \right| \leq \frac{2}{a^3} \left[ \left(1 - \frac{2h}{ķ}\right) \left\{ \frac{\left(1 - \frac{2h}{ķ}\right)^2 (\dot{c}-a)^2}{24} + \frac{1}{2}(L-A)^2 \right\} + \frac{h^3 (\dot{c}-a)^2}{3ķ^3} \right].$$

**Remark 3.3.**
Consider $g:(0,\infty)\to R$ such that

$$g(x)=lnx,\ \ x\in\left[a+h.\frac{\dot{c}-a}{ķ},\dot{c}-h.\frac{\dot{c}-a}{ķ}\right]\subset(0,\infty)$$

then

$$\frac{1}{\dot{c}-a}\int_{a}^{\dot{c}}g(x)\,dx=ln\ I(a,\dot{c})$$

$$g(a)+g(\dot{c})=ln\ G^{2}(a,\dot{c}),\qquad g'(\dot{c})-g'(a)=-\frac{\dot{c}-a}{G^{2}(a,\dot{c})}$$

$$||\,g''||_{\infty}=\frac{1}{a^{2}}$$

So, (1) becomes

$$\begin{aligned}&\left|\left(1-\frac{2h}{ķ}\right)\left[lnx-1+\frac{A}{x}\right]+\frac{h}{ķ}\left[ln\ G^{2}+\frac{h(\dot{c}-a)^{2}}{2ķG^{2}}\right]-ln\ I\right|\\&\le\frac{(\dot{c}-a)^{2}}{24a^{2}}\left[3\left(1-\frac{2h}{ķ}\right)^{2}+1\right].\end{aligned}\tag{8}$$

Choosing $x=A(a,\dot{c})$ in (8), we get

$$\begin{aligned}&\left|ln\ \frac{A^{\left(1-\frac{2h}{ķ}\right)}}{I}+\frac{h}{ķ}\left[ln\ G^{2}+\frac{h(\dot{c}-a)^{2}}{2ķG^{2}}\right]\right|\\&\le\frac{(\dot{c}-a)^{2}}{24a^{2}}\left[3\left(\frac{2h}{ķ}-\frac{1}{2}\right)^{2}+\frac{1}{4}\right]\end{aligned}$$

Choosing $x=I(a,\dot{c})$ in (8), we get

$$\begin{aligned}&\left|\frac{2h}{ķ}ln\ I+\left(1-\frac{2h}{ķ}\right)\left[1-\frac{A}{I}\right]-\frac{h}{ķ}\left[ln\ G^{2}+\frac{h(\dot{c}-a)^{2}}{2ķG^{2}}\right]\right|\\&\le\frac{1}{a^{2}}\left[\left(1-\frac{2h}{ķ}\right)\left\{\frac{\left(1-\frac{2h}{ķ}\right)^{2}(\dot{c}-a)^{2}}{24}+\frac{1}{2}(I-A)^{2}\right\}+\frac{h^{3}(\dot{c}-a)^{2}}{3ķ^{3}}\right].\end{aligned}$$

**Remark 3.4.**
Consider the mapping $g:(0,\infty)\to R$ such that

$$g(x)=x^{r},r\in R\backslash\{-1,0\}$$

then

$$\begin{aligned}&\frac{1}{\dot{c}-a}\int_{a}^{\dot{c}}g(x)\,dx=L_{r}^{r}(a,\dot{c})\\&g(a)+g(\dot{c})=2A\left(a^{r},\dot{c}^{r}\right),\qquad g'(\dot{c})-g'(a)=r(r-1)(\dot{c}-a)L_{r-2}^{r-2}(a,\dot{c})\\&||\,g''||_{1}=|r(r-1)|(\dot{c}-a)L_{r-1}^{r-1}(a,\dot{c})\end{aligned}$$

So, the inequality (1) gives

$$\begin{aligned}&\left|\left(1-\frac{2h}{\text{ķ}}\right)\left[x^{r}-r\left(x-A\right)x^{r-1}\right]\right.\\&\left.+\frac{2h}{\text{ķ}}\left[A\left(a^{r},\dot{c}^{r}\right)-\frac{hr\left(r-1\right)\left(\dot{c}-a\right)^{2}}{4\text{ķ}}L_{r-2}^{r-2}\right]-L_{r}^{r}\right|\\&\leq\frac{\left|r\left(r-1\right)\right|\left(\dot{c}-a\right)^{2}}{2}\left(1-\frac{2h}{\text{ķ}}\right)^{2}L_{r-1}^{r-1}\end{aligned}\tag{9}$$

Choosing $x=A\left(a,\dot{c}\right)$in ((9), we get

$$\begin{aligned}&\left|\left(1-\frac{2h}{\text{ķ}}\right)A^{r}+\frac{2h}{\text{ķ}}\left[A\left(a^{r},\dot{c}^{r}\right)-\frac{hr\left(r-1\right)\left(\dot{c}-a\right)^{2}}{4\text{ķ}}L_{r-2}^{r-2}\right]-L_{r}^{r}\right|\\&\leq\frac{\left|r\left(r-1\right)\right|\left(\dot{c}-a\right)^{2}}{8}\left(1-\frac{2h}{\text{ķ}}\right)^{2}L_{r-1}^{r-1}\end{aligned}$$

**Remark 3.5.**
Consider $g:(0,\infty)\to R$ such that

$$g\left(x\right)=\frac{1}{x},\ x\in\left[a+h.\frac{\dot{c}-a}{\text{ķ}},\dot{c}-h.\frac{\dot{c}-a}{\text{ķ}}\right]\subset(0,\infty)$$

then

$$\frac{1}{\dot{c}-a}\int_{a}^{\dot{c}}g\left(x\right)dx=L^{-1}\left(a,\dot{c}\right)$$

$$g\left(a\right)+g\left(\dot{c}\right)=\frac{2}{H\left(a,\dot{c}\right)}=\frac{2A\left(a,\dot{c}\right)}{G^{2}\left(a,\dot{c}\right)},\qquad g'\left(\dot{c}\right)-g'\left(a\right)=\frac{2\left(\dot{c}-a\right)A\left(a,\dot{c}\right)}{G^{4}\left(a,\dot{c}\right)}$$

$$\left\|g''\right\|_{1}=2\left(\dot{c}-a\right)L_{-3}^{-3}\left(a,\dot{c}\right)$$

So, the inequality (1) gives

$$\begin{aligned}&\left|\left(1-\frac{2h}{\text{ķ}}\right)\left[2-\frac{A}{x}\right]\frac{1}{x}+\frac{2hA}{\text{ķ}G^{2}}\left[1-\frac{h(\dot{c}-a)^{2}}{2\text{ķ}G^{2}}\right]-L^{-1}\right|\\&\leq\left(1-\frac{2h}{\text{ķ}}\right)^{2}\left(\dot{c}-a\right)^{2}L_{-3}^{-3}\end{aligned}\tag{10}$$

Choosing $x=A\left(a,\dot{c}\right)$ in (10) , we get

$$\begin{aligned}&\left|\left(1-\frac{2h}{\text{ķ}}\right)\frac{1}{A}+\frac{2hA}{\text{ķ}G^{2}}\left[1-\frac{h(\dot{c}-a)^{2}}{2\text{ķ}G^{2}}\right]-L^{-1}\right|\\&\leq\frac{\left(\dot{c}-a\right)^{2}}{4}\left(1-\frac{2h}{\text{ķ}}\right)^{2}L_{-3}^{-3}\end{aligned}$$

Choosing $x=L\left(a,\dot{c}\right)$ in (10) , we get

$$\begin{aligned}&\left|\left[\left(1-\frac{2h}{\text{ķ}}\right)\left(2-\frac{A}{L}\right)-1\right]\frac{1}{L}+\frac{2hA}{\text{ķ}G^{2}}\left[1-\frac{h(\dot{c}-a)^{2}}{2\text{ķ}G^{2}}\right]\right|\\&\leq\left[\left|L-A\right|+\frac{1}{2}\left(1-\frac{2h}{\text{ķ}}\right)\left(\dot{c}-a\right)\right]^{2}L_{-3}^{-3}\end{aligned}$$

**Remark 3.6.**
Consider $g:(0,\infty)\to R$ such that

$$g(x)=lnx,\ x\in\left[a+h.\frac{\dot{c}\ -\ a}{ķ},\dot{c}-h.\frac{\dot{c}\ -\ a}{ķ}\right]\subset(0,\infty)$$

then

$$\frac{1}{\dot{c}-a}\int_{a}^{\dot{c}}g(x)\,dx=ln\ I(a,\dot{c})$$

$$g(a)+g(\dot{c})=ln\ G^{2}(a,\dot{c}),\quad g'(\dot{c})-g'(a)=-\frac{\dot{c}\ -\ a}{G^{2}(a,\dot{c})},$$

$$||\ g''||_{1}=(\dot{c}-a)\,L_{-2}^{-2}(a,\dot{c})$$

So, the inequality (1), gives

$$\left|\left(1-\frac{2h}{ķ}\right)\left[lnx-1+\frac{A}{x}\right]+\frac{h}{ķ}\left[ln\ G^{2}+\frac{h(\dot{c}-a)^{2}}{2ķG^{2}}\right]-ln\ I\right| \le\frac{(\dot{c}-a)^{2}}{2}\left(1-\frac{2h}{ķ}\right)^{2}L_{-2}^{-2} \tag{11}$$

Choosing $x=A(a,\dot{c})$ in (11), we get

$$\left|ln\ \frac{A^{\left(1-\frac{2h}{ķ}\right)}}{I}+\frac{h}{ķ}\left[ln\ G^{2}+\frac{h(\dot{c}-a)^{2}}{2ķG^{2}}\right]\right| \le\frac{(\dot{c}-a)^{2}}{8}\left(1-\frac{2h}{ķ}\right)^{2}L_{-2}^{-2}$$

Choosing $x=I(a,\dot{c})$ in (11), we get

$$\left|\frac{2h}{ķ}ln\ I+\left(1-\frac{2h}{ķ}\right)\left[1-\frac{A}{I}\right]-\frac{h}{ķ}\left[ln\ G^{2}+\frac{h(\dot{c}-a)^{2}}{2ķG^{2}}\right]\right| \le\frac{1}{2}\left[|I-A|+\frac{1}{2}\left(1-\frac{2h}{ķ}\right)(\dot{c}-a)\right]^{2}L_{-2}^{-2}.$$

# 4. Application for Numerical Integration-I

Let $I_{n}:\ a=u_{0}<u_{1}<u_{2}<...<u_{n-1}<u_{n}=\dot{c}$ be a dissection of $[a,\dot{c}]$, $\varsigma_{i}\in\left[u_{ĭ}+\delta.\frac{h_{i}}{ķ},u_{ĭ+1}-\delta.\frac{h_{i}}{ķ}\right]$ be a sequence of intermediate points, $h_{i}=u_{ĭ+1}-u_{ĭ}$ ( $i=0,1....,n-1$ ), then we get:

**Theorem 4.1.**
*Let $g:[a,\dot{c}]\to R$ be a twice differentiable on $(a,\dot{c})$, with $g''\in L_{\infty}(a,\dot{c})$*

$$i.e.\ ||\ g''||_{\infty}=\underset{t\in(a,\dot{c})}{Sup}\ |\ g''(t)|<\infty,$$

*then*

$$\int_a^{\dot{c}} g(u)\,du = A\left(g, g', I_n, \varsigma, \delta\right) + R\left(g, g', I_n, \varsigma, \delta\right),$$

*where*

$$\begin{aligned} & A\left(g, g', I_n, \varsigma, \delta\right) \\ = & \left(1 - \frac{2\delta}{ķ}\right) \sum_{i=0}^{n-1} \left[g\left(\varsigma_i\right) - \left(\varsigma_i - \frac{u_{\check{\imath}} + u_{\check{\imath}+1}}{2}\right) g'\left(\varsigma_i\right)\right] h_i \\ & + \frac{\delta}{ķ} \sum_{i=0}^{n-1} \left(g\left(u_{\check{\imath}}\right) + g\left(u_{\check{\imath}+1}\right)\right) h_i - \frac{\delta^2}{2ķ^2} \sum_{i=0}^{n-1} h_i^2 \Delta g'\left(u_{\check{\imath}}\right) \end{aligned}$$

*and the reminder* $R\left(g, g', I_n, \varsigma, \delta\right)$ *satisfies the estimation*

$$\begin{aligned} & \left|R\left(g, g', I_n, \varsigma, \delta\right)\right| \\ \leq & \left[3\left(1 - \frac{2\delta}{ķ}\right)^2 + 1\right] \left|\left|\, g''\right|\right|_\infty \sum_{i=0}^{n-1} \frac{h_i^3}{24} \end{aligned}$$

*Proof.* By using Theorem 2.1 on $[u_{\check{\imath}}, u_{\check{\imath}+1}]$, $(i = 0, 1, 2, ..., n-1)$, we obtain:

$$\begin{aligned} & \left|\left(1 - \frac{2\delta}{ķ}\right)\left[\, g(\varsigma_i) - \left(\varsigma_i - \frac{u_{\check{\imath}} + u_{\check{\imath}+1}}{2}\right) g'(\varsigma_i)\right] h_i \right. \\ & \left. + \frac{\delta}{ķ}\left[\left(g(u_{\check{\imath}}) + g\left(u_{\check{\imath}+1}\right)\right) - \frac{h_i \delta}{2ķ} \Delta g'(u_{\check{\imath}})\right] h_i - \int_{u_{\check{\imath}}}^{u_{\check{\imath}+1}} g(t)dt \right| \\ \leq & \left[3\left(1 - \frac{2\delta}{ķ}\right)^2 + 1\right] \frac{h_i^3}{24} \left|\left|\, g''\right|\right|_\infty \end{aligned}$$

Imply $\sum_{i=0}^{n-1}$ and with the help of triangular inequality, we get the desired inequality. □

**Corollary 4.1.**
*The following perturbed mid-point rule holds:*

$$\int_a^{\dot{c}} g(u)\,du = A_M\left(g, g', I_n, \delta\right) + R_M\left(g, g', I_n, \delta\right)$$

*where*

$$\begin{aligned} & A_M\left(g, g', I_n, \delta\right) \\ = & \left(1 - \frac{2\delta}{ķ}\right) \sum_{i=0}^{n-1} g\left(\frac{u_{\check{\imath}} + u_{\check{\imath}+1}}{2}\right) h_i \\ & + \frac{\delta}{ķ} \sum_{i=0}^{n-1} \left(g\left(u_{\check{\imath}}\right) + g\left(u_{\check{\imath}+1}\right)\right) h_i - \frac{\delta^2}{2ķ^2} \sum_{i=0}^{n-1} h_i^2\ \Delta g'\left(u_{\check{\imath}}\right) \end{aligned}$$

*and the reminder term* $R_M\left(\, g, g', I_n, \delta\,\right)$ *satisfies the estimation*

$$\begin{aligned} & \left|R_M\left(g, g', I_n, \delta\right)\right| \\ \leq & \left[3\left(\frac{2\delta}{ķ} - \frac{1}{2}\right)^2 + \frac{1}{4}\right] \left|\left|\, g''\right|\right|_\infty \sum_{i=0}^{n-1} \frac{h_i^3}{24} \end{aligned}$$

Again, the following perturbed trapezoidal rule holds:

$$\int_a^{\dot{c}} g(u)\, du = A_T\left(g, g', I_n, \delta\right) + R_T\left(g, g', I_n, \delta\right),$$

where

$$A_T\left(g, g', I_n, \delta\right) = \sum_{i=0}^{n-1}\left(\frac{g(u_{\breve{\imath}}) + g(u_{\breve{\imath}+1})}{2}\right) h_i - \frac{1}{4}\left[\left(1-\frac{\delta}{\c{k}}\right)^2 + \left(\frac{\delta}{\c{k}}\right)^2\right]\sum_{i=0}^{n-1} h_i^2 \Delta g'(u_{\breve{\imath}})$$

and $R_T\left(g, g', I_n, \delta\right)$ satisfies the estimation

$$\left|R_T\left(g, g', I_n, \delta\right)\right| \le \left[3\left(1-\frac{2\delta}{\c{k}}\right)^2 + 1\right] \|g''\|_\infty \sum_{i=0}^{n-1}\frac{h_i^3}{24}.$$

## 5. Application for Numerical Integration-II

By using same idea, that we already used above, we get the following quadrature formula:

Let $g : [a, \dot{c}] \to R$ be continuous on $[a, \dot{c}]$ and twice differentiable on $(a, \dot{c})$ such that $g'' \in L_1(a, \dot{c})$

$$i.e. \|g''\|_1 = \int_a^{\dot{c}} |g''(t)|\, dt$$

then we have

$$\int_a^{\dot{c}} g(u) du = A\left(g, g', I_n, \varsigma, \delta\right) + R\left(g, g', I_n, \varsigma, \delta\right)$$

where

$$A\left(g, g', I_n, \varsigma, \delta\right) = \left(1-\frac{2\delta}{\c{k}}\right)\sum_{i=0}^{n-1}\left[g(\varsigma_i) - \left(\varsigma_i - \frac{u_{\breve{\imath}} + u_{\breve{\imath}+1}}{2}\right) g'(\varsigma_i)\right] h_i + \frac{\delta}{\c{k}}\sum_{i=0}^{n-1}\left(g(u_{\breve{\imath}}) + g(u_{\breve{\imath}+1})\right) h_i - \frac{\delta^2}{2\c{k}^2}\sum_{i=0}^{n-1} h_i^2 \Delta g'(u_{\breve{\imath}})$$

and $R(g, g', I_n, \varsigma, \delta)$ satisfies the estimation

$$\left|R\left(g, g', I_n, \varsigma, \delta\right)\right| \le \frac{1}{2}\left[\frac{1}{2}\left(1-\frac{2\delta}{\c{k}}\right)\nu(h) + \sup_{i=0,1,\dots n-1}\left|\varsigma_i - \frac{u_{\breve{\imath}} + u_{\breve{\imath}+1}}{2}\right|\right]^2 \|g''\|_1 \le \frac{\nu^2(h)}{2}\left(1-\frac{2\delta}{\c{k}}\right)^2 \|g''\|_1.$$

where $\nu(h) = \max\{u_{\check{\imath}+1} - u_{\check{\imath}} \mid i = 0, 1, ..., n-1\}$

*Proof.* Apply Theorem 2.1 on $[u_{\check{\imath}}, u_{\check{\imath}+1}]$, ( $i = 0, 1, 2, ..., n-1$ ), we obtain:

$$\begin{aligned}
&\left|\left(1-\frac{2\delta}{\c{k}}\right)\left[g(\varsigma_i) - \left(\varsigma_i - \frac{u_{\check{\imath}}+u_{\check{\imath}+1}}{2}\right)g'(\varsigma_i)\right]h_i\right. \\
&\left.+\frac{\delta}{\c{k}}\left[(g(u_{\check{\imath}}) + g(u_{\check{\imath}+1})) - \frac{h_i\delta}{2\c{k}}\Delta g'(u_{\check{\imath}})\right]h_i - \int\limits_{u_{\check{\imath}}}^{u_{\check{\imath}+1}} g(t)dt\right| \\
&\leq \frac{1}{2}\left(1-\frac{2\delta}{\c{k}}\right)(u_{\check{\imath}+1} - u_{\check{\imath}})\int\limits_{u_{\check{\imath}}}^{u_{\check{\imath}+1}} |g(t)|\,dt
\end{aligned}$$

By using same technique, as we already used above, we get the desired inequality. □

## 6. Application for Numerical Integration-III

Again using same idea, that we already used above, we get the following quadrature formula:

Let $g : [a, \dot{c}] \to R$ be a twice differentiable on $(a, \dot{c})$ such that $g'' \in L_p(a, \dot{c}), p > 1$

$$i.e. \,||\, g''||_p = \left(\int\limits_a^{\dot{c}} |\, g''(t)|^p \,dt\right)^{\frac{1}{p}}$$

then we have

$$\int\limits_a^{\dot{c}} g(u)du = A\left(g, g', I_n, \varsigma, \delta\right) + R\left(g, g', I_n, \varsigma, \delta\right)$$

where

$$\begin{aligned}
&A\left(g, g', I_n, \varsigma, \delta\right) \\
&= \left(1-\frac{2\delta}{\c{k}}\right)\sum_{i=0}^{n-1}\left[g(\varsigma_i) - \left(\varsigma_i - \frac{u_{\check{\imath}}+u_{\check{\imath}+1}}{2}\right)g'(\varsigma_i)\right]h_i \\
&+\frac{\delta}{\c{k}}\sum_{i=0}^{n-1}(g(u_{\check{\imath}}) + g(u_{\check{\imath}+1}))h_i - \frac{\delta^2}{2\c{k}^2}\sum_{i=0}^{n-1}h_i^2\Delta g'(u_{\check{\imath}})
\end{aligned}$$

and the reminder $R(\, g, g', I_n, \varsigma, \delta\,)$satisfies the estimation

$$\begin{aligned}
&\left|R\left(g, g', I_n, \varsigma, \delta\right)\right| \\
&\leq \frac{1}{2(2q+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{n-1}\left[\left(1-\frac{2\delta}{\c{k}}\right)^{2q+1} + 2\left(\frac{\delta}{\c{k}}\right)^{2q+1}\right]h_i^{2q+1}\right)^{\frac{1}{q}} ||\, g''||_p
\end{aligned}$$

*Proof.* Apply Theorem 2.1 on $[u_{\breve{\imath}}, u_{\breve{\imath}+1}]$, $(\ i = 0, 1, 2, ..., n-1\ )$, we obtain:

$$\Bigg|\left(1-\frac{2\delta}{\c{k}}\right)\left[\,g(\varsigma_i)-\left(\varsigma_i-\frac{u_{\breve{\imath}}+u_{\breve{\imath}+1}}{2}\right)g'(\varsigma_i)\right]h_i$$
$$+\frac{\delta}{\c{k}}\left[\left(g(u_{\breve{\imath}})+g\left(u_{\breve{\imath}+1}\right)\right)-\frac{h_i\delta}{2\c{k}}\Delta g'(u_{\breve{\imath}})\right]h_i-\int_{u_{\breve{\imath}}}^{u_{\breve{\imath}+1}}g(t)dt\Bigg|$$
$$\leq\frac{1}{2\left(2q+1\right)^{\frac{1}{q}}}\left(\left[\left(1-\frac{2\delta}{\c{k}}\right)^{2q+1}+2\left(\frac{\delta}{\c{k}}\right)^{2q+1}\right]h_i^{2q+1}\right)^{\frac{1}{q}}||\,g''||_p$$

By using same technique, as we already used above, we get the desired inequality. □

# 7. conclusion

We established generalized Ostrowski type inequality for differentiable mappings whose second derivative belongs to different Lebesgue spaces as like $L_\infty[a,b]$, $L_1[a,b]$ and $L_p[a,b]$. Here we show that the inequalities obtained in [5], [8], [10], [23] and [24] are special cases of our inequalities. Applications are also discussed.

# Competing Interests

There are no competing interests.

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