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Average no regret sentinel

Research Article

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Abstract: In this paper, we consider a wave equation with pollution term that includes a fractional derivative of Riemann-Liouville with missing Newman boundary condition and the operator depending on unknown parameter. First we try to avoid the unknown parameter and to isolate the missing function the results are established by introduce the notion of average no regret sentinel. Second is applied the classical approach of sentinel to identify the pollution.

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1. Introduction

When we modeling almost natural phenomena, we cannot access or know all the information related to the phenomenon that causes us to fall into the problem with incomplete data, this missing can be in the source terms initial or boundary conditions or parameters in the main operators.

In environmental or ecology problems, we never know the initial data a real example is pollution phenomena. Our basic goal is to use the Sentinel method Lions [11] to obtain information about the pollution terms (source items) independent to the unknown terms in the initial condition. This method needs for a state equation, some observation function, and some control function to be determined. This method is based on the following steps, first we prove the equivalence between the existence of sentinel and null-controllability problems, then we solve the last problems by Carleman inequality and lax-Milgram or penalization method. The key tool used to identified the pollution terms is Taylor development, in several works using this notion see the following papers (see [13], [14] and [17]).

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In different phenomena like physical or biomedical the missing information holds in the initial or boundary condition or in the second member of the equation from the situation in this case leads to use the notion of no-regret control presented by Lions 1992 [10] related by a sequence of low regret control which converges weakly to the no regret control and assists us with getting the optimality systems that characterize both controls. As of late, Zuazua in 2014 [28] introduced the notion of average control to solve the problem governed by unknown parameters in the main operators.

In light of the absence of two kinds of missing data for example unknown velocity of propagation and unknown Dirichlet boundary condition in electromagnetic wave equation describing a biomedical phenomenon prompts the utilization of the notion of average no regret control which was introduced in [7]. Moreover, the average Sentinel [26] is used to identified the pollution terms in the heat equation which the state relies on unknown parameters. However, in this paper, we are interested to find an estimation or identified a pollution terms of a wave equation that penetrate a polluted medium with missing physical information. The unknown velocity of propagation is represented by an unknown datum, likewise the flux is missing which represent by a Newman boundary condition, this makes us in a complicated problem that contains all the types of incomplete data. To solve her we represent the notion of average no-regret Sentinel. We follow the same way in (see [7] and [26]), first we applied the average sentinel then we use the idea of no regret control finally we applied the classical sentinel method.

Our paper is organized as follows, in the first section we give some fundamental definition concerning the fractional calculus then we describe our problem which given by fractional equation represented by Riemann-Liouville sense. In the second we applied the notion of average no regret Sentinel, a third section we applied the classical Sentinel method to get information of the pollution terms, we finish by conclusion.

2. Preliminaries

2.1. Fractional calculus

Fractional calculus appeared in 1695 when Leibniz wrote to Hospital "Can the meaning of derivatives with integer-order be generalized to derivatives with non-integer orders?" L'Hopital posed another question to Leibniz : "What if the order will be 1/2?". Leibniz replied: "It will lead to a paradox, from which one day useful consequences will be drawn". Since that time many scientists have tried to give the answer to this question until Lacroix [8] gave the correct answer in 1819. until now, the number of publications of fractional calculus has been growing, attracting many scientists to many branches of its discipline (physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc), the secret of the big issues of fractional calculus is that the fractional-order models are often more approximated to real-world systems and more accurate rather the classical calculus. in last decade, many formulas and definitions have been developed such as the Grunwald-Letnikov, Caputo and Riemann-Liouvilles approaches, For more details on the development of fractional calculus see [9]. In particular we suggest the Riemann-Liouvilles definitions.

Definition 2.1.
(Riemann–Liouville integral) The Riemann–Liouville fractional integral of order α is defined by the following expression :

$$I_{+}^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} f(s) \, ds.$$

Definition 2.2.
(Riemann–Liouville derivative) The Riemann–Liouville fractional derivative of order α is defined by the following expression :

$$D_{+}^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{0}^{t}(t-s)^{-\alpha} f'(s) \, ds.$$

Where $f: R^{+} \rightarrow R$ be a continuous function on R^{+}, $t>0$ and $0<\alpha<1$. $\Gamma(.)$ is the Gamma function defined for any complex number z by

$$\Gamma(z)=\int_{\infty}^{0} t^{z-1} \exp^{-t} dt.$$

Remark 2.1.
([8]) For $n-1<\alpha<n$:

$$D_{+}^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{0}^{t}(t-s)^{-\alpha} f'(s) \, ds.$$

Lemma 2.1.
([3],[15],[16]) Let $\phi \in C^{\infty}(\bar{Q})$, we have

$$\begin{aligned}
\int_{0}^{T} \int_{\Omega}\left(D_{+}^{\alpha} y(x, t)+\mathcal{A}(t) y(x, t)\right) \phi(x, t) \, dx dt = & \int_{0}^{T} \int_{\Omega} y(x, t)\left(-D^{\alpha} \phi(x, t)\right) \\
& +\mathcal{A}^{*}(t) \phi(x, t) \, dx dt+\int_{\Omega} \phi(x, T) I_{+}^{1-\alpha} y(x, T) \, dx \\
& -\int_{\Omega} \phi\left(x, 0^{+}\right) I_{+}^{1-\alpha} y\left(x, 0^{+}\right) dx \\
& +\int_{0}^{T} \int_{\Gamma}\left(y \frac{\partial \phi}{\partial \nu}-\phi \frac{\partial y}{\partial \nu}\right) d\Gamma dt.
\end{aligned}$$

Where $\mathcal{A}(t)$ is a given operator, $I_{+}^{1-\alpha} y\left(x, 0^{+}\right)=\lim_{t \rightarrow 0^{+}} I_{+}^{1-\alpha} y(x, t)$ and $-D^{\alpha}$ is the so-called right fractional Riemann–Liouville derivative given by

$$D_{+}^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t}^{T}(s-t)^{-\alpha} f'(s) \, ds.$$

2.2. Model describing

Let us consider the fractional diffusion wave equation that penetrates a polluted medium with missing information. we set $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$, where Ω is an open bounded domain in R^n $(n = 1,2,3)$ with smooth boundary Γ, $t \in (0,T)$, $T > 0$

$$\begin{cases} D_+^{\alpha} y - \sigma^2 \Delta y &= \xi + \lambda \hat{\xi} & \text{in } Q, \\ I_+^{1-\alpha} y\left(0^+\right) &= y_0 + \tau \hat{y}_0 & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} &= g & \text{on } \Sigma, \end{cases} \tag{1}$$

where $y = y\left(x,t,\sigma,g,\lambda,\tau\right) \in L^2\left(Q\right)$, $\sigma \in [\sigma_1, \sigma_2]$ represent the velocity of propagation, g is an unknown function belongs to $L^2\left(\Sigma\right)$. The function $\xi = \xi\left(x,t\right) \in L^2\left(Q\right)$ and $y_0 = y_0\left(x\right) \in L^2\left(\Omega\right)$ are known but the functions $\left\{\lambda\hat{\xi}, \tau\hat{y}_0\right\}$ are unknown and reprint the pollution term and perturbation term resp, $\hat{\xi}$ & $\hat{y}_0$ are renormalized and represent the size of pollution and perturbation

$$\left\|\hat{\xi}\right\|_{L^2(Q)} \leq 1, \ \ \left\|\hat{y}_0\right\|_{L^2(\Omega)} \leq 1,$$

with λ, τ are small enough.

There are three types of incomplete data for this problem. The first is the pollution term we want to identify, the second is the missing parameter σ, and the last is the unknown function g.

We aim at choosing a control independent of the value of both the unknowns (the parameters and the function) to determine the pollution terms $\lambda\hat{\xi}$ independent to the unknown $\tau\hat{y}_0$.

3. Presentation of the method

In order to identify the pollution term is to use the sentinel method. In our case, applying the classical sentinel approach of Lions gives us information related to the missing $\{\sigma,\ g\}$, we have two obstacles to avoid them we assume to take the average of the state then we applied the idea of no regret control.

Let the control function $w \in L^2\left(O \times (0,T)\right)$ and an observer $h_0 \in L^2\left(O \times (0,T)\right)$ we define the functional :

$$s\left(w,\sigma,g,\lambda,\tau\right) = \int_0^T \int_\Omega \left(h_0 \chi_O + w \chi_O\right) y\left(w,\sigma,g,\lambda,\tau\right) dx dt, \tag{2}$$

where O is an observatory domain and open and non empty subsets of Ω.

Now, we need to eliminate the datum σ and isolate g that we will explain now :

First, we assume to take the average of the state y respect to the datum σ then the functional (2) becomes as

$$s\left(w,g,\lambda,\tau\right) = \int_0^T \int_\Omega \left(h_0 \chi_O + w \chi_O\right) \left(\int_{\sigma_1}^{\sigma_2} y\left(w,g,\sigma,\lambda,\tau\right) d\sigma\right) dx dt, \tag{3}$$

where $\int\limits_{\sigma_1}^{\sigma_2} y(\lambda,\tau,\sigma,g)$ is the average of the state. Now we seems to take only the control satisfying

$$s(w,g,\lambda,\tau) \leq s(0,g,\lambda,\tau), \ \forall g \in L^2(\Sigma).$$

Lemma 3.1.
For all $w \in L^2(O \times (0,T))$ and $g \in L^2(\Sigma)$ we have

$$s(w,g,\lambda,\tau) - s(0,g,\lambda,\tau) = \int_0^T \int_\Omega (h_0\chi_O + w\chi_O)\,\Psi dxdt,$$

and $\Psi = \int\limits_{\sigma_1}^{\sigma_2} (y(w,g,\sigma,\lambda,\tau) - y(0,0,\sigma,\lambda,\tau))\,d\sigma.$

Proof. We know that

$$y(w,g,\sigma,\lambda,\tau) = y(w,0,\sigma,\lambda,\tau) + y(0,g,\sigma,\lambda,\tau) - y(0,0,\sigma,\lambda,\tau). \tag{4}$$

Where $y(w,0,\sigma,\lambda,\tau)$, $y(0,g,\sigma,\lambda,\tau)$, $y(w,0,\sigma,\lambda,\tau)$ are respectively solutions of

$$\begin{cases} D_+^\alpha y - \sigma^2 \Delta y &= \xi + \lambda\hat{\xi} & \text{in } Q, \\ I_+^{1-\alpha} y(0^+) &= y_0 + \tau\hat{y}_0 & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} &= 0 & \text{on } \Sigma, \end{cases}$$

$$\begin{cases} D_+^\alpha y - \sigma^2 \Delta y &= \xi + \lambda\hat{\xi} & \text{in } Q, \\ I_+^{1-\alpha} y(0^+) &= y_0 + \tau\hat{y}_0 & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} &= g & \text{on } \Sigma, \end{cases}$$

$$\begin{cases} D_+^\alpha y - \sigma^2 \Delta y &= \xi + \lambda\hat{\xi} & \text{in } Q, \\ I_+^{1-\alpha} y(0^+) &= y_0 + \tau\hat{y}_0 & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} &= 0 & \text{on } \Sigma, \end{cases}$$

replace (4) in (2), we get

$$\begin{aligned} s(w,g,\lambda,\tau) - s(0,g,\lambda,\tau) &= \int_0^T \int_\Omega (h_0\chi_O + w\chi_O) \\ &\times \left(\int_{\sigma_1}^{\sigma_2} (y(w,0,\sigma,\lambda,\tau) + y(0,g,\sigma,\lambda,\tau) - y(0,0,\sigma,\lambda,\tau))\,d\sigma \right) dxdt \\ &- \int_0^T \int_\Omega (h_0\chi_O + w\chi_O) \left(\int_{\sigma_1}^{\sigma_2} (y(0,g,\sigma,\lambda,\tau))\,d\sigma \right) dxdt \\ &= \int_0^T \int_\Omega (h_0\chi_O + w\chi_O) \left(\int_{\sigma_1}^{\sigma_2} (y(w,0,\sigma,\lambda,\tau) - y(0,0,\sigma,\lambda,\tau))\,d\sigma \right) dxdt. \end{aligned}$$

□

Remark 3.1.
We noted that $\varsigma = \varsigma(w,\lambda,\tau) = \int\limits_{\sigma_1}^{\sigma_2} \left(y(w,0,\sigma,\lambda,\tau) - y(0,0,\sigma,\lambda,\tau)\right) d\sigma$ solution of

$$\left\{ \begin{array}{rcl} D_+^\alpha \varsigma - \sigma^2 \Delta \varsigma &=& 0 \ \text{ in } Q, \\ I_+^{1-\alpha} \varsigma\left(0^+\right) &=& 0 \ \text{ in } \Omega, \\ \frac{\partial \varsigma}{\partial \nu} &=& 0 \ \text{ on } \Sigma, \end{array} \right. \tag{5}$$

We can't get any information for the pollution term $\{\lambda\hat{\xi}\}$ with working by systems (3), for this reason we relax our problem by take

$$s(w,g,\lambda,\tau) \leq s(0,g,\lambda,\tau) + s(0,0,\lambda,\tau), \ \forall g \in L^2(\Sigma).$$

We obtain :
$s(w,g,\lambda,\tau) - s(0,g,\lambda,\tau) - s(0,0,\lambda,\tau) = \int\limits_0^T \int\limits_\Omega (h_0\chi_O + w\chi_O)$
$\times \left(\int\limits_{\sigma_1}^{\sigma_2} \left(y(w,0,\sigma,\lambda,\tau) - 2y(0,0,\sigma,\lambda,\tau)\right) d\sigma \right) dxdt.$
We suppose that $\eta(w,\sigma,\lambda,\tau) = y(w,0,\sigma,\lambda,\tau) - 2y(0,0,\sigma,\lambda,\tau)$ solution of

$$\left\{ \begin{array}{rcll} D_+^\alpha \eta - \sigma^2 \Delta \eta &=& -\left(\xi + \lambda\hat{\xi}\right) & \text{in } Q, \\ I_+^{1-\alpha} \eta\left(0^+\right) &=& -\left(y_0 + \tau \hat{y}_0\right) & \text{in } \Omega, \\ \frac{\partial \eta}{\partial \nu} &=& 0 & \text{on } \Sigma, \end{array} \right. \tag{6}$$

then

$$S(w,\lambda,\tau) = \int\limits_0^T \int\limits_\Omega (h_0\chi_O + w\chi_O) \left(\int\limits_{\sigma_1}^{\sigma_2} \eta(w,\sigma,\lambda,\tau)\, d\sigma \right) dxdt,$$

supposed that

$$t = \sigma t \Rightarrow \sigma = 1,$$

$$\int\limits_{\sigma_1}^{\sigma_2} \eta(w,\sigma,\lambda,\tau,x,t)\, d\sigma = \int\limits_{\sigma_1 t}^{\sigma_2 t} \eta(w,1,\lambda,\tau,x,\sigma t) \frac{dt}{t} = z(x,t),$$

where $z(x,t)$ solution of

$$\left\{ \begin{array}{rcll} D_+^\alpha z - \Delta z &=& \Xi + \lambda\hat{\Xi} & \text{in } Q, \\ I_+^{1-\alpha} z\left(0^+\right) &=& y_0 + \tau \hat{y}_0 & \text{in } \Omega, \\ \frac{\partial z}{\partial \nu} &=& 0 & \text{on } \Sigma, \end{array} \right. \tag{7}$$

with

$$\Xi = \int\limits_{\sigma_2 t}^{\sigma 1 t} \frac{\xi}{t} dt, \ \hat{\Xi} = \int\limits_{\sigma_2 t}^{\sigma_1 t} \frac{\hat{\xi}}{t} dt,$$

$$z_0 = \int\limits_{\sigma_2 t}^{\sigma_1 t} \frac{y_0}{t} dt, \ \int\limits_{\sigma_2 t}^{\sigma_1 t} \frac{y_0}{t} dt.$$

The Sentinel functional becomes as

$$S(\lambda,\tau)=\int_0^T\int_\Omega (h_0\chi_O+w\chi_O)\,z(x,t,\lambda,\tau)\,dxdt.$$

Which is our classical Sentinel problem.

4. Classical approach of Sentinel method

In this section, we can applied the classical approach of Sentinel method, when we prove the existence of a sentinel equivalent to null controllability problem and we use penalization to solve our problem. Consequently, the estimates of pollution terms arise when we use development of Taylor.

Definition 4.1.
We say that S defines a sentinel for the problem (7) if there exists w such that S is insensitive with respect the to missing terms $\tau\hat{z}_0$ i.e.,

$$\left.\frac{\partial S}{\partial \tau}(\lambda,\tau)\right|_{(\lambda,\tau)=(0,0)}=0;\ \text{for all } \hat{z}_0, \tag{8}$$

$$w\in L^2(O\times(0,T)),\text{of minimal norm.} \tag{9}$$

Theorem 4.1.
The condition (8) leads to a null-controllability of the adjoint problem q

$$\begin{cases} -D^\alpha q-\Delta q &= h_0\chi_O+w\chi_O & \textit{in } Q,\\ q(T) &= 0 & \textit{in } \Omega,\\ \frac{\partial q}{\partial\nu} &= 0 & \textit{on } \Sigma,\end{cases} \tag{10}$$

with

$$q(0)=0\ \textit{in } \Omega.$$

Proof. The insensibility condition (8) holds iff

$$\int_0^T\int_\Omega (h_0\chi_O+w\chi_O)\,z_\tau dxdt=0,$$

where $z_\tau=\left.\frac{\partial z}{\partial\tau}(\lambda,\tau)\right|_{(\lambda,\tau)=(0,0)}$ solution of

$$\begin{cases} D_+^\alpha z_\tau-\Delta z_\tau &= 0 & \text{in } Q,\\ I_+^{1-\alpha}z_\tau\left(0^+\right) &= \hat{z}_0 & \text{in } \Omega,\\ \frac{\partial z_\tau}{\partial\nu} &= 0 & \text{on } \Sigma,\end{cases} \tag{11}$$

Then, we introduce the adjoint state q given by (10). □

In other side, we multiply the first equation in (11) by q and integrate over Q,

$$\begin{aligned}\int_0^T\int_\Omega \left(D_+^\alpha z_\tau - \Delta z_\tau\right) q\,dxdt &= \int_0^T\int_\Omega \left(-D^\alpha q - \Delta q\right) z_\tau\,dxdt \\ &+ \int_\Omega q\left(T\right) I_+^{1-\alpha} z_\tau\left(T\right) dx - \int_\Omega q\left(0^+\right) I_+^{1-\alpha} z_\tau\left(0^+\right) dx \\ &+ \int_0^T\int_\Gamma \left(z_\tau \frac{\partial q}{\partial \nu} - q\frac{\partial z_\tau}{\partial \nu}\right) d\Gamma dt \\ &= \int_0^T\int_\Omega \left(h_0\chi_O + w\chi_\omega\right) z_\tau\,dxdt - \int_\Omega q\left(0^+\right) z_\tau dx = 0.\end{aligned}$$

We deduce

$$\int_\Omega q\left(0^+\right) z_\tau dx = 0.$$

Finally

$$q\left(0^+\right) = 0 \text{ in } \Omega.$$

4.1. Resolution of null-controllability problem

we devise the problem (10) into two problems

$$\begin{cases} -D^\alpha q_0 - \Delta q_0 &= h_0\chi_O \quad \text{in } Q, \\ q_0\left(T\right) &= 0 \quad \text{in } \Omega, \\ \frac{\partial q_0}{\partial \nu} &= 0 \quad \text{on } \Sigma, \end{cases} \tag{12}$$

$$\begin{cases} -D^\alpha z - \Delta z &= w\chi_O \quad \text{in } Q, \\ z\left(T\right) &= 0 \quad \text{in } \Omega, \\ \frac{\partial z}{\partial \nu} &= 0 \quad \text{on } \Sigma, \end{cases} \tag{13}$$

then $q = q_0 + z$, where q_0 is known, we seeks to check the control w which z satisfying

$$z\left(0\right) = -q_0\left(0\right).$$

4.2. Penalization and system of optimality

Theorem 4.2.
The couple (w, z) is characterized by the following optimality systems

$$\begin{cases} -D^\alpha z - \Delta z &= \rho\chi_O \quad \textit{in } Q, \\ \frac{\partial z}{\partial \nu} &= 0 \quad \textit{on } \Sigma, \\ z\left(T\right) &= 0 \quad \textit{in } \Omega, \end{cases}$$

$$\begin{cases} D_{+}^{\alpha}\rho - \Delta\rho &= 0 \quad in\ Q, \\ \rho &= 0 \quad on\ \Sigma, \\ \rho(0) &= \rho^{0} \quad in\ \Omega, \end{cases}$$

with

$$w = \chi_{O}\rho.$$

For $\epsilon > 0$, we introduce the following functional

$$J_{\epsilon}(w_{\epsilon}, z) = \frac{1}{2}\|w_{\epsilon}\|^{2}_{L^{2}(O\times(0,T))} + \frac{1}{2\epsilon}\|-D^{\alpha}z - \Delta z - w\chi_{O}\|^{2}_{L^{2}(\Omega\times(0,T))},$$

we choose all z verifying

$$-D^{\alpha}z - \Delta z - w\chi_{O} \in L^{2}(\Omega\times(0,T)),$$
$$\frac{\partial z}{\partial\nu} = 0 \text{ on } \Sigma,$$
$$z(T) = 0 \text{ in } \Omega,$$
$$z(0) = -q_{0}(0),$$

let $(w_{\epsilon}, z_{\epsilon})$ solution of

$$\inf J_{\epsilon}(w_{\epsilon}, z),$$

supposed that

$$\rho_{\epsilon} = \frac{1}{\epsilon}\left(-D^{\alpha}z_{\epsilon} - \Delta z_{\epsilon} - w_{\epsilon}\chi_{\omega}\right),$$

then

$$J_{\epsilon}(w_{\epsilon})(w_{\epsilon} - \hat{w}) = 0,\ \forall w_{\epsilon} \in L^{2}(O\times(0,T)),$$

we get

$$(w_{\epsilon}, \hat{w})_{L^{2}(O\times(0,T))} + (\rho_{\epsilon}, -D^{\alpha}\hat{z} - \Delta\hat{z} - \hat{w}\chi_{O})_{L^{2}(\Omega\times(0,T))} = 0, \tag{14}$$

such that

$$-D^{\alpha}\hat{z} - \Delta\hat{z} - \hat{w}\chi_{O} \in L^{2}(\Omega\times(0,T)),$$
$$\frac{\partial\hat{z}}{\partial\nu} = 0 \text{ on } \Sigma,$$
$$\hat{z}(0) = 0,\ \hat{z}(T) = 0 \text{ in } \Omega,$$

we deduce that

$$D_{+}^{\alpha}\rho_{\epsilon} - \Delta\rho_{\epsilon} = 0 \text{ in } Q,$$
$$\rho_{\epsilon} = 0 \text{ on } \Sigma.$$

Without any information about $\rho_\epsilon(0)$ and $\rho_\epsilon(T)$.

Replace the last equation in (14), we get

$$(w_\epsilon - \chi_O \rho_\epsilon, \hat{w})_{L^2(O\times(0,T))} = 0,$$

it means that

$$w_\epsilon = \chi_O \rho_\epsilon.$$

In a suitable topology, we suppose that $\rho_\epsilon \to \rho$ when $\epsilon \to 0$, we get

$$\begin{cases} D_+^\alpha \rho - \Delta\rho &= 0 \quad \text{in } Q, \\ \rho(0) &= \rho^0 \quad \text{in } \Omega, \\ \rho &= 0 \quad \text{on } \Sigma, \end{cases}$$

also,

$$\begin{cases} -D^\alpha z - \Delta z &= \rho\chi_O \quad \text{in } Q, \\ \frac{\partial z}{\partial \nu} &= 0 \quad \text{on } \Sigma, \\ z(T) &= 0 \quad \text{in } \Omega, \end{cases} \tag{15}$$

with

$$w = \chi_O \rho.$$

In other side, we search ρ^0 in way that

$$z(0) = -q_0(0).$$

Multiplying (13) by ρ we obtain after integrating by part and help of (15)

$$\int_0^T \int_\Omega (-D^\alpha z - \Delta z)\rho dx dt = \int_0^T \int_o \chi_O \rho^2 dx dt$$

$$\int_\Omega z(0)\rho^0 dx = \int_0^T \int_O \rho^2 dx dt,$$

we define an operator

$$\Lambda\rho^0 = z(0),$$

we obtain

$$(\Lambda\rho^0, \rho^0) = \int_0^T \int_O \rho^2 dx dt,$$

we introduce

$$\|\rho^0\|_F = \left(\int_0^T \int_O \rho^2 dx dt\right)^{\frac{1}{2}}. \tag{16}$$

One indicates F the space of Hilbert separate and supplemented regular functions ρ^0 for the norm (16).
$\Lambda \in L\left(F, F^{'}\right)$ is an isomorphism of F to $F^{'}$ ($F^{'}$dual space of F), and $\Lambda = \Lambda^*$.

$$\Lambda \rho^0 = -q_0\left(0\right),$$

implies that

$$\rho^0 = -\Lambda^{-1} q_0\left(0\right),$$

it's enough to show that

$$q_0\left(0\right) \in F^{'},$$

multiply (12) by ρ, we get

$$\left(q_0\left(0\right), \rho^0\right) = \int\limits_0^T \int\limits_\Omega \left(h_0 + \rho\right) \chi_O dxdt,$$

thus

$$\left|\left\langle q_0\left(0\right), \rho^0\right\rangle\right| \leq \left\|h_0\right\|_{L^2(O\times(0,T))} + \left\|\rho\right\|_F,$$

we deduce

$$\left\|q_0\left(0\right)\right\|_{F'} \leq \left\|h_0\right\|_{L^2(O\times(0,T))},$$

then

$$S\left(\lambda, \tau\right) = \int\limits_0^T \int\limits_\Omega \left(h_0 + \rho\right) \chi_O z\left(\lambda, \tau\right) dxdt.$$

4.3. Identification of the Pollution Term

We denote an observation of z by

$$z_{obs} = z\chi_O = m_0,$$

then the measured sentinel associate to z_{obs} is given by

$$S_{obs}\left(\lambda, \tau\right) = \int\limits_0^T \int\limits_\Omega \left(h_0 + \rho\right) \chi_O m_0 dxdt.$$

Theorem 4.3.
The pollution term is to estimate as follows

$$\int\limits_0^T \int\limits_\Omega \lambda \hat{\Xi} q dxdt \simeq \int\limits_0^T \int\limits_\Omega \left(h_0 + \rho\right) \chi_O \left(m_0 - z_0\right) dxdt.$$

Using development of Taylor, we get

$$S_{obs}\left(\lambda,\tau\right)\simeq S\left(0,0\right)+\lambda\frac{\partial S}{\partial\lambda}\left(0,0\right)+\tau\frac{\partial S}{\partial\tau}\left(0,0\right),$$

due to (8)

$$S_{obs}\left(\lambda,\tau\right)\simeq S\left(0,0\right)+\lambda\frac{\partial S}{\partial\lambda}\left(0,0\right).$$

Hence,

$$\lambda\frac{\partial S}{\partial\lambda}\left(0,0\right)\simeq S_{obs}\left(\lambda,\tau\right)-S\left(0,0\right), \tag{17}$$

in other hand, we have

$$\frac{\partial S}{\partial\lambda}\left(0,0\right)=\int\limits_0^T\int\limits_\Omega\left(h_0+\rho\right)\chi_O z_\lambda dxdt, \tag{18}$$

where $z_\lambda=\frac{\partial z}{\partial\lambda}\left(\lambda,\tau\right)\big|_{(\lambda,\tau)=(0,0)}$ solution of the following

$$\begin{cases} D_+^\alpha z_\lambda-\Delta z_\lambda &= \hat{\Xi} \text{ in } Q,\\ I_+^{1-\alpha}z_\lambda\left(0^+\right) &= 0 \text{ in } \Omega,\\ \frac{\partial z_\lambda}{\partial\nu} &= 0 \text{ on } \Sigma,\end{cases} \tag{19}$$

Multiply the first equation by q and integrate over Q

$$\int\limits_0^T\int\limits_\Omega\left(D_+^\alpha z_\lambda-\Delta z_\lambda\right)qdxdt=\int\limits_0^T\int\limits_\Omega\hat{\Xi}qdxdt,$$

in view to Lemma (4) we get the following equations

$$\begin{aligned}\int\limits_0^T\int\limits_\Omega z_\lambda\left(-D_+^\alpha q-\Delta q\right)dxdt&=\int\limits_0^T\int\limits_\Omega\left(h_0\chi_O+w\chi_\omega\right)z_\lambda dxdt\\&=\int\limits_0^T\int\limits_\Omega\hat{\Xi}qdxdt,\end{aligned}$$

thank's to (12) and (13), we get

$$\int\limits_0^T\int\limits_\Omega\lambda\hat{\Xi}qdxdt\simeq\int\limits_0^T\int\limits_\Omega\left(h_0+\rho\right)\chi_O\left(m_0-z_0\right)dxdt.$$

5. Conclusion

In this work we present the average no-regret Sentinel to estimate the pollution term in fractional diffusion wave equation when the state governed by unknown datum and missing Newman boundary condition when the classical approach of Sentinel method gives us information related to the missing data for this we try to avoided this problems by combine the notion of average control and the idea of no-regret control. This method can be also used in border Sentinel and weakly Sentinel.

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