Innovative Journal of Mathematics
v. 1 no. 2, 44-62, (2022)
doi:10.55059/ijm.2022.1.2/26

ISSN 2790-3370 (print)
ISSN 2790-3389 (online)

Sentinel method and distributed systems with missing data

Research Article

Merabti Nesrine Lamya[1]*, Imad Rezzoug[1]†

1 Laboratory of Dynamics systems and control, University of Oum El Bouaghi, Algeria

Abstract: In this work, we study an approximate controllability problem with constraint on the control. This problem appears naturally of weak and pointwise sentinel with a small or singleton region observation. The main tool is a theorem of uniqueness of the solution of ill-posed Cauchy problem for the parabolic equation.

MSC: 93B05, 35K20, 49J20

Keywords: Parabolic equation • Approximate controllability • Pointwise and Weak sentinel • Pollution term

Received 2022-02-06; Accepted 2022-03-08; Published 2022-04-14

1. Notion of the sentinel

The study of dynamic of spatiotemporal systems has generated wide literature with applications in many fields as such ecology, immunology, desertification, population dynamics, pollution as well as many others. Interesting problem for such systems concerns incomplete data and state measurement on a certain region of its geometric domain. In the case of distributed systems defined on a geometric domain Ω, numerous papers were devoted to the state controllability in the whole domain Ω (see Lions[10, 12] and the references therein). This work caters with regional analysis paradigm developed by Zerrik [31], El Jai [7] and others, by using the weakly sentinel notion introduced by Rezzoug and Ayadi [1, 20] for pollution estimation where the measurement region $\mathcal{O}$ is either Ω or in the pointwises of Ω. Precisely, we consider a parabolic distributed parameter system defined on the geometric domain Ω and we assume that the following assumptions are given :

- An open regular and bounded set Ω of $\mathbb{R}^n$, $n \geq 1$,with boundary $\Gamma = \partial\Omega$.

† **Corresponding author**

* *E-mail:* *merabtimerabti2015@gmail.com*
† *E-mail:* *imad.rezzoug@univ-oeb.dz*

- A time interval $[0,T]$ we denote $\mathcal{Q} =]0,T[\ \times\Omega$ and $\Sigma =]0,T[\ \times\partial\Omega$.

- A second order differential linear operator A with compact resolvent and which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the state space $X = L^2(\Omega)$.

- A^* will denote the adjoint operator of A.

Then the considered system which is described by the following state equations

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) + Ay(t,x) &= F(t,x) \quad \text{in } \mathcal{Q}, \\ y(0,x) &= y_0(x) \quad \text{in } \Omega, \\ y(t,x) &= g(t,x) \quad \text{on } \Sigma, \end{cases} \tag{1}$$

where

$$F \in L^2(\mathcal{Q}), g \in L^2(\Sigma) \text{ and } y_0 \in L^2(\Omega).$$

Have unique weak solution. And we note that

$$\int_\Omega yAqdx - \int_\Omega qA^*ydx = \int_\Gamma y\frac{\partial}{\partial v_A}qd\Gamma - \int_\Gamma q\frac{\partial}{\partial v_{A*}}yd\Gamma,$$

for all y and q in the Sobolev space $H^1(\Omega)$. In systems theory, the sentinel is related to the possibility of finding the state of the adjoint system dynamics independently of the missing and pollution terms, and of the choose of control spaces. The regional (boundary) sentinel explores the notion of sentinel in the particular case where the support of the initial state of adjoint system dynamics is into the subregion (a part of boundary) ω.

2. Regional sentinel

In this section, we choose $\mathcal{O}$ in the interior of Ω and we assume that the considered system is described by the following equations

$$\begin{cases} \frac{\partial y}{\partial t}(t,x;\lambda,\tau) + Ay(t,x;\lambda,\tau) &= f_0(t,x) + \lambda f(t,x) \quad \text{in } \mathcal{Q}, \\ y(0,x;\lambda,\tau) &= y_0(x) + \tau\bar{y}(x) \quad \text{in } \Omega, \\ y(t,x;\lambda,\tau) &= 0 \quad \text{on } \Sigma, \end{cases} \tag{2}$$

where f_0, y_0 are given ; f, y are unknown functions and λ, τ are small unknown parameters. Let h_0 be a function given in $L^2((0,T)\times\mathcal{O})$. One considers a functional defined by the formula

$$S(\lambda,\tau) = \int\int_{(0,T)\times\mathcal{O}} (h_0+\varphi)y(x,t;\lambda,\tau)dxdt, \tag{3}$$

where $\varphi \in L^2((0,T)\times\mathcal{O})$.

Definition 2.1.
The functional $S(\lambda,\tau)$ is said to be regional sentinel defined by h_0 if the following properties are satisfied :
1) there exists $u \in L^2([0,T]\times\mathcal{O})$ such that $\frac{\partial S}{\partial\tau}(\lambda,\tau)_{\lambda=0,\tau=0} = 0$, for all $\bar{y} \in L^2([0,T]\times\Omega)$ such that its spacial support is into $\mathcal{O}$.
2) $\|u\| = \inf\|\varphi\|$ for all φ satisfying the property one.

Now we focus on the regional sentinel construction : let $\widehat{y}(t,x)$ be the unique solution of the following equations

$$\begin{cases} \frac{\partial \widehat{y}}{\partial t}(t,x;0,0) + A\widehat{y}(t,x;0,0) &= f_0(t,x) \quad \text{in } \mathcal{Q}, \\ \widehat{y}(0,x;0,0) &= y_0(x) \quad \text{in } \Omega, \\ \widehat{y}(t,x;0,0) &= 0 \quad \text{on } \Sigma. \end{cases} \tag{4}$$

The derivative of the system (2) with respect to the parameter λ near $(\lambda = 0, \tau = 0)$ is given by the following equations

$$\begin{cases} \frac{\partial y_\lambda}{\partial t}(t,x) + Ay_\lambda(t,x) &= f(t,x) \quad \text{in } \mathcal{Q}, \\ y_\lambda(0,x;\lambda,\tau) &= 0 \quad \text{in } \Omega, \\ y_\lambda(t,x) &= 0 \quad \text{on } \Sigma, \end{cases} \tag{5}$$

and also the derivative of the system (2) with respect to the parameter τ near $(\lambda = 0, \tau = 0)$ is given by the following equations

$$\begin{cases} \frac{\partial y_\tau}{\partial t}(t,x) + Ay_\tau(t,x) &= 0 \quad \text{in } \mathcal{Q}, \\ y_\tau(0,x) &= \bar{y} \quad \text{in } \Omega, \\ y_\tau(t,x) &= 0 \quad \text{on } \Sigma. \end{cases} \tag{6}$$

The adjoint system associated to (6) is defined by the following equations

$$\begin{cases} -\frac{\partial q}{\partial t}(t,x) + A^*q(t,x) &= (h_0(t,x) + \varphi(t,x))\chi_{\mathcal{O}}(x) \quad \text{in } \mathcal{Q}, \\ q(T) &= 0 \quad \text{in } \Omega, \\ q &= 0 \quad \text{on } \Sigma, \end{cases} \tag{7}$$

with h_0 and φ in $L^2(]0,T[\times\mathcal{O})$. The system (7) is decomposed into two systems, free one and forced one. The free system is given by the following equations

$$\begin{cases} -\frac{\partial q_0}{\partial t}(t,x) + A^*q_0(t,x) &= h_0(t,x)\chi_{\mathcal{O}}(x) \quad \text{in } \mathcal{Q}, \\ q_0(T) &= 0 \quad \text{in } \Omega, \\ q_0 &= 0 \quad \text{on } \Sigma, \end{cases} \tag{8}$$

the forced system is given by the following equations

$$\begin{cases} -\frac{\partial q_1}{\partial t}(t,x) + A^*q_1(t,x) &= \varphi(t,x)\chi_{\mathcal{O}}(x) \quad \text{in } \mathcal{Q}, \\ q_1(T) &= 0 \quad \text{in } \Omega, \\ q_1 &= 0 \quad \text{on } \Sigma, \end{cases} \tag{9}$$

then the solution of (7) is written as

$$q = q_0 + q_1.$$

Definition 2.2.
The dynamic system (9) is said to be regionally controllable on the region ω if, for all desired state, there exists a control such that the final state is equal to the considered desired state on ω.

We consider $q_0(0,.) \in L^2(\Omega)$ as the desired state and we take a region $\omega = \Omega \backslash \mathcal{O}$. Then the regional controllability consists in finding a control u in $L^2(]0,T[\,; L^2(\mathcal{O}))$ which permits, in a finite time, to bring the state q_1 of system (9) from the initial state $q_1(T,x) = 0$, to the final desired state $-q_0(0,x)$ on this region.

Remark 2.1.
If the higher multiplicity of the eigenvalue of A is equal to one, then the system (9) is controllable in $L^2(\omega)$, [7, 31].

Theorem 2.1.
If the system (9) is ω regional controllable, then there exists a unique control $u \in L^2(]0,T[\,; L^2(\mathcal{O}))$ which satisfies the definition 2.1 of the sentinel.

Proof. If the system (9) is regionally controllable on ω then, for $q_0(0)$ is given in $L^2(\mathcal{O})$, there exists a unique control $u \in L^2((0,T)\times\mathcal{O})$ such that $q_1(0)\chi_\omega = -q_0(0)\chi_\omega$, hence we get the first formula of the definition 2.1. From the equation (6) and the equation (7) we can deduce

$$-\int_{\mathcal{O}} q(0)\bar{y}dx = \int\int_{(0,T)\times\mathcal{O}} (h_0+u)y_\tau(x,t;\lambda,\tau)dxdt, \tag{10}$$

and hence, for any $\bar{y}$ having its support outside $\mathcal{O}$, we have $\int_{\mathcal{O}} q(0)\bar{y}dx = 0$, hence

$$\frac{\partial}{\partial\tau}S(\lambda,\tau)_{\lambda=0,\tau=0} = \int\int_{(0,T)\times\mathcal{O}} (h_0+u)y_\tau(x,t;\lambda,\tau)dxdt = 0. \tag{11}$$

□

2.1. Estimate of the pollution terms

Now, Let $y_m(t,x)$ be the measured state of the system on the observatory $\mathcal{O}$ during the interval $]0,T[$, then the measured regional sentinel is given by formula

$$S_m(\lambda,\tau) = \int\int_{(0,T)\times\mathcal{O}} (h_0+u)y_m(x,t;\lambda,\tau)dxdt. \tag{12}$$

Theorem 2.2.
If the system (9) is $\omega-$regionally controllable then we have the following estimation

$$\int_{[0,T]\times\omega} qf dxdt = S_m(\lambda,\tau) - S(0,0).$$

Proof. We know that

$$S(\lambda,\tau)=S(0,0)+\lambda\frac{\partial}{\partial\lambda}S(\lambda,\tau)_{\lambda=0,\tau=0}\ +\tau\frac{\partial}{\partial\tau}S(\lambda,\tau)_{\lambda=0,\tau=0}, \tag{13}$$

using the equations (11) and (12) we have

$$S_m(\lambda,\tau)-S(0,0)=\lambda\frac{\partial}{\partial\lambda}S(\lambda,\tau)_{\lambda=0,\tau=0}, \tag{14}$$

where

$$\frac{\partial}{\partial\lambda}S(\lambda,\tau)\,|_{\lambda=0,\tau=0}\ =\int\int_{\mathcal{O}\times(0,T)}(h_0+u)y_\lambda(x,t)dxdt, \tag{15}$$

and

$$S(0,0)=\int\int_{\mathcal{O}\times(0,T)}(h_0+u)\widehat{y}(x,t)dxdt, \tag{16}$$

using the equations (5) and (7), we deduce that

$$\int\int_{(0,T)\times\mathcal{O}}(h_0+u)y_\lambda(x,t)dxdt=\int_{(0,T)\times\Omega}qfdx,$$

hence

$$\lambda\int_{(0,T)\times\Omega}qfdx=S_m(\lambda,\tau)-S(0,0).$$

□

3. Pointwise sentinel

In this section, we choose $\mathcal{O}=\{b\}$ a point in Ω and we assume that the considered system is described by the equation (2). Let h_0 be a function given in $L^2(0,T)$,one considers a functional defined by the formula

$$S(\lambda,\tau)=\int_0^T(h_0(t)+\varphi(t))y(b,t;\lambda,\tau)dt, \tag{17}$$

where $\varphi\in L^2(0,T)$.

Definition 3.1.
The functional $S(\lambda,\tau)$ is said to be pointwise sentinel defined by h_0 if the following properties are satisfied :
1) there exists $u\in L^2([0,T])$ such that $\frac{\partial S}{\partial\tau}(\lambda,\tau)_{\lambda=0,\tau=0}$=0 for all $\bar{y}\in L^2([0,T]\times\Omega)$ and $\bar{y}$ =0 on $[0,T]\times\{b\}$.
2) $\|u\|=\inf\|\varphi\|$ for all φ satisfying the property 1).

Now we focus on the Pointwise sentinel construction: let $\widehat{y}(t,x)$ be the some solution of (4),the derivative solution with respect to the parameter λ is given by (5) and also the derivative solution with respect to the parameter τ is given by (5) The adjoint system associated to (5) is defined by the following equations

$$\begin{cases}-\frac{\partial q}{\partial t}(t,x)+A^*q(t,x) &= (h_0(t)+\phi(t))\delta(b-x) \quad \text{in } \mathcal{Q},\\ q(T,x) &= 0 \quad \text{in } \Omega,\\ q &= 0 \quad \text{on } \Sigma,\end{cases} \tag{18}$$

with h_0 and φ in $L^2(0,T)$. The system (18) is decomposed into two systems, free one and forced one. The free system is given by the following equations

$$\begin{cases} -\frac{\partial q_0}{\partial t}(t,x) + A^* q_0(t,x) &= h_0(t)\delta(b-x) \quad \text{in } \mathcal{Q}, \\ q_0(T,x) &= 0 \quad \text{in } \Omega, \\ q_0 &= 0 \quad \text{on } \Sigma. \end{cases} \tag{19}$$

The forced system is given by the following equations

$$\begin{cases} -\frac{\partial q_1}{\partial t}(t,x) + A^* q_1(t,x) &= \varphi(t)\delta(b-x) \quad \text{in } \mathcal{Q}, \\ q_1(T,x) &= 0 \quad \text{in } \Omega, \\ q_1 &= 0 \quad \text{on } \Sigma, \end{cases} \tag{20}$$

there is one function φ such that

$$q_1 \Big|_{\Omega/\{b\}} (t,x) = -q_0 \Big|_{\Omega/\{b\}} (t,x) u(t,x) = \varphi(t)\delta(b-x)$$

hence

$$q \Big|_{\Omega/\{b\}} (t,x) = 0 \text{ and } u(t,x) = \varphi(t)\delta(b-x)$$

$$q = q_0 + q_1.$$

Multiplying the equation (6) by q and integrating by parts, we have

$$\int_0^T f(b,t) q(b,t) dt = \int_0^T (h_0 + u) y_\tau dt = 0$$

Let $y_m(t,x)$ be a measured state of the system on the observatory $\{b\}$ during the interval $]0,T[$, then the measured sentinel is given by

$$S_m(\lambda,\tau) = \int_0^T (h_0 + u) y_m(t,b;\lambda,\tau) dt, \tag{21}$$

and we write

$$S(\lambda,\tau) = S(0,0) + \lambda \frac{\partial}{\partial \lambda} S(\lambda,\tau)_{\lambda=0,\tau=0} + \tau \frac{\partial}{\partial \tau} S(\lambda,\tau)_{\lambda=0,\tau=0}, \tag{22}$$

where

$$S(0,0) = \int_0^T (h_0 + u)(t) \widehat{y}(t,b) dt. \tag{23}$$

3.1. Estimate of the pollution terms

In this section, the objective is to estimate the pollution terms independently of the missing terms.

Theorem 3.1.
Under the hypothesis of the theorem 2.1, the pollution term of the system (5) is estimated independently of the missing term by

$$S_m(\lambda,\tau) - S(0,0) = \int_0^T (h(t)+u(t))(y_m(t,b) - \hat{y}(t,b))dt,$$

where $\hat{y}$ is the solution of (4) and y_m is the observed state in $\{b\}$ during the time interval $[0,T]$.

Proof. Let $S(\lambda,\tau)$ be the sentinel defined by h_0, from the equation (22), we can deduce :

$$\lambda \frac{\partial}{\partial \lambda} S(\lambda,\tau)\,|_{\lambda=0,\tau=0} = S(\lambda,\tau) - S(0,0),$$

as we know that $S(\lambda,\tau) = S_m(\lambda,\tau)$ in the point $\{b\}$, then we deduce from the equations (17) and (18) :

$$\frac{\partial}{\partial \lambda} S(\lambda,\tau)\,|_{\lambda=0,\tau=0} = \int_0^T (h_0+u)y_\lambda(t,b)dt = \int_0^T qf(t,b)dt,$$

thus

$$\begin{aligned} \lambda \int_0^T q(t,b)f(t,b)dxdt &= S_m(\lambda,\tau) - S(0,0) \\ &= \int_0^T (h_0+u)(y_m(t,b) - \widehat{y}(t,b))dt. \end{aligned}$$

□

4. Weak sentinel

4.1. Formulation problem

For $n = \{2;3\}$, let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$ of class $\mathcal{C}^2$, $T > 0$, and let $\mathcal{O}$ be an open non empty subset of Ω. Set $\mathcal{Q} = (0,T)\times\Omega$, $\Sigma = (0,T)\times\Gamma$, $\mathcal{U} = (0,T)\times\mathcal{O}$. We consider the parabolic equation :

$$\left\{ \begin{array}{rclll} y' - \Delta y + f(y) &=& \xi + \lambda\widehat{\xi} & \text{in} & \mathcal{Q}, \\ y(0) &=& y_0 + \tau\widehat{y}_0 & \text{in} & \Omega, \\ y &=& 0 & \text{on} & \Sigma. \end{array} \right. \tag{24}$$

Where $(.)'$ is the partial derivative with respect to time t.

Remark 4.1.
The problem (24) admits a unique solution. For the sake of simplicity, we denote $y(x,t;\lambda,\tau) = y(\lambda,\tau)$.
One supposes that the data ξ is rather regular, and that the terms of pollution "that one wants to estimate" are rather regular. It will be always supposed that the solution y check at least $y \in L^2(\mathcal{Q})$.

Remark 4.2.
One will always indicate by y_0 the solution $y\left(x,t;0,0\right);$ thus

$$\left\{\begin{array}{rclll} y_0' - \Delta y_0 + f\left(y_0\right) & = & \xi & \text{in} & \mathcal{Q}, \\ y_0 & = & 0 & \text{on} & \Sigma, \\ y_0(0) & = & y_0 & \text{in} & \Omega. \end{array}\right. \tag{25}$$

The problem considered here consists in trying to estimate $\lambda\widehat{\xi}$ starting from observations, distributed or borders, without seeking to estimate the term lack $\tau\widehat{y}_0$.

One starts with a distributed observation, therefore a distributed sentinel.

4.2. The weak sentinel method

Definition 4.1.
(definition, existence and uniqueness of the sentinel)
Let $h \in L^2(\mathcal{U})$ and for any control function $u \in L^2(\mathcal{U})$, set

$$S\left(\lambda,\tau\right) = \int_{\mathcal{Q}} \left(h+u\right)\chi_{\mathcal{O}}\, y\left(x,t;\lambda,\tau\right) dxdt, \tag{26}$$

the functional S is said to be weak sentinel if it satisfies the following conditions :
for all $\epsilon > 0$ there exists $u \in L^2(\mathcal{U})$ such as

$$u \in L^2(\mathcal{U}),\ \text{of minimal norm.} \tag{27}$$

$$\left|\frac{\partial}{\partial\tau} S\left(0,0\right)\right| \le \epsilon. \tag{28}$$

Remark 4.3.
The function $u = -h$ give place to (26) so that the problem (27, 28) admits a single solution, which is defined by h.
The problem is thus :
(1) to calculate this solution ;
(2) to see whether the corresponding sentinel justifies its name, i.e. gives information on pollution $\lambda\widehat{\xi}$.

4.2.1. Adjoint state

The adjoint state is introduced q by

$$\left\{\begin{array}{rcll} -q' - \Delta q + f'\left(y_0\right) q & = & \left(h+u\right)\chi_{\mathcal{O}} & \text{in } \mathcal{Q}, \\ q & = & 0 & \text{on } \Sigma, \\ q\left(T\right) & = & 0 & \text{in } \Omega. \end{array}\right. \tag{29}$$

Where $(.)'$ is the partial derivative with respect to time t, $h, u \in L^2\left(\mathcal{U}\right)$.

Remark 4.4.
System (29) is the adjoint parabolic problem. It appears under this form in J.L.Lions sentinels theory as the associated adjoint state.

We multiply (29) by y_τ and we integrate by parts, we have

$$\left(q\left(0\right),\widehat{y}_0\right)=\int_{\mathcal{Q}}\left(h+u\right)\chi_{\mathcal{O}}y_\tau\left(x,t;\lambda,\tau\right)dxdt,$$

y_τ is defined by

$$\begin{cases} y_\tau'-\Delta y_\tau+f'\left(y_0\right)y_\tau & = & \widehat{\xi} & \text{in} & \mathcal{Q}, \\ y_\tau\left(0\right) & = & \widehat{y}_0 & \text{in} & \Omega, \\ y_\tau & = & 0 & \text{on} & \Sigma. \end{cases}$$

So we get

$$\frac{\partial}{\partial\tau}\mathcal{S}\left(0,0\right)=\left(q\left(0\right),\widehat{y}_0\right), \tag{30}$$

so that (28) is equivalent to

$$\left\|q\left(x,0\right)\right\|_{L^2(\Omega)}\leq\epsilon. \tag{31}$$

There is thus business with a problem of the type "approximate controllability with zero" .

4.2.2. The main result

The main result is the following

Lemma 4.1.
Let $v\in L^2\left(\mathcal{U}\right)$. Then there is no $\rho\in L^2\left(\mathcal{Q}\right),\rho\neq0$ such that ρ satisfies

$$\begin{cases} -\rho'-\Delta\rho+f'\left(y_0\right)\rho & = & 0 \text{ in } \mathcal{Q}, \\ \rho & = & 0 \text{ on } \Sigma, \\ \rho\chi_{\mathcal{O}} & = & v. \end{cases} \tag{32}$$

Proof. If the problem (32) admits a solution, then it is given by

$$\rho\left(x,t\right)=\sum_{j=1}^{\infty}\alpha_j\left(t\right)u_j\left(x\right). \tag{33}$$

Where u_j are eigenfunctions of

$$\begin{cases} -\Delta u & = & \lambda u & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma. \end{cases} \tag{34}$$

Differentiate the solution (34) once with respect to t and twice with respect to x and substitute these derivatives into the first equation of (32). We then obtain

$$\sum_{j=1}^{\infty}\left(\alpha_j'\left(t\right)\lambda_j\alpha_j\left(t\right)\right)u_j\left(x\right)=0. \tag{35}$$

Thus,

$$\alpha_j'\left(t\right)-\lambda_j\alpha_j\left(t\right)=0. \tag{36}$$

Because (u_j) form an orthonormal base of $L^2(\mathcal{Q})$. Furthermore, the function ρ satisfies the boundary conditions if and only if

$$\sum_{j=1}^{\infty} \alpha_j(t)\, u_j(x) = v\chi_{\mathcal{O}}. \tag{37}$$

As $v\chi_{\mathcal{O}} \in L^2(\mathcal{Q})$ then

$$v\chi_{\mathcal{O}} = \sum_{j=1}^{\infty} \langle v\chi_{\mathcal{O}}, u_j\rangle_{L^2(\mathcal{Q})} u_j(x). \tag{38}$$

Consequently

$$\alpha_j(t) = \langle v\chi_{\mathcal{O}}, u_j\rangle_{L^2(\mathcal{Q})}. \tag{39}$$

Finally, we have

$$\begin{cases} -\alpha_j'(t) + \lambda_j \alpha_j(t) &= 0 \text{ in } (0,T), \\ \alpha_j(t) &= \langle v\chi_{\mathcal{O}}, u_j\rangle_{L^2(\mathcal{Q})}. \end{cases} \tag{40}$$

Then the solution of the first order linear is given by

$$\alpha_j(t) = \langle v\chi_{\mathcal{O}}, u_j\rangle_{L^2(\mathcal{Q})} e^{\lambda_j t}. \tag{41}$$

Consequently, if the problem (32) admits a solution, it is necessarily in the form :

$$\rho(x,t) = \sum_{j=1}^{\infty} \langle v\chi_{\mathcal{O}}, u_j\rangle_{L^2(\mathcal{Q})} e^{\lambda_j t} u_j(x). \tag{42}$$

We prove now that $\rho \notin L^2(\mathcal{Q})$. Indeed,

$$\begin{aligned} \int_0^T |\alpha_j(t)|^2 dt &= \left|\langle v\chi_{\mathcal{O}}, u_j\rangle_{L^2(\mathcal{Q})}\right|^2 \int_0^T e^{2\lambda_j t} dt \\ &= \left|\langle v\chi_{\mathcal{O}}, u_j\rangle_{L^2(\mathcal{Q})}\right|^2 \left[\frac{-1}{2\lambda_j} + \frac{1}{2\lambda_j} e^{2\lambda_j T}\right]. \end{aligned} \tag{43}$$

But, λ_j is the eigenvalue of problem (34), then $\lambda_j \underset{j\longmapsto\infty}{\longrightarrow} \infty$. Consequently,

$$\int_0^T |\alpha_j(t)|^2 dt \underset{j\longmapsto\infty}{\longrightarrow} \infty. \tag{44}$$

Which means that the series whose general term $\alpha_j(t)$ is not normally convergent. So, problem (32) admits no solution. □

Theorem 4.1.
For $\epsilon > 0$, $h \in L^2(\mathcal{U})$, there exists some control u and some state q such that (29) and (31) hold. Moreover, there exists a unique pair $(\widehat{u}\chi_{\mathcal{O}}, \widehat{q})$ with $\widehat{u}$ of minimal norm in $L^2(\mathcal{U})$, i.e. such that (29, 31) and (27) hold.

Proof. Let q be a solution of the system (29) and q_0 a solution of the following system

$$\left\{\begin{array}{rcll} -q_0' - \Delta q_0 + f'(y_0)\, q_0 & = & h\chi_{\mathcal{O}} & \text{in } \mathcal{Q}, \\ q_0 & = & 0 & \text{on } \Sigma, \\ q_0(T) & = & 0 & \text{in } \Omega. \end{array}\right. \tag{45}$$

We put

$$q = q_0 + z \tag{46}$$

Then, z is the solution of the following problem

$$\left\{\begin{array}{rcll} -z' - \Delta z + f'(y_0)\, z & = & u\chi_{\mathcal{O}} & \text{in } \mathcal{Q}, \\ z & = & 0 & \text{on } \Sigma, \\ z(T) & = & 0 & \text{in } \Omega. \end{array}\right. \tag{47}$$

We now introduce the set of states reachable at time 0 defined by

$$\mathcal{F}(0) = \left\{ z(u,0) \text{ such as } u \in L^2(\mathcal{U}) \right\}. \tag{48}$$

It is clear that $\mathcal{F}(0)$ is a vector subspace of $L^2(\Omega)$. According to the **HAHN-BANACH** theorem, it will be dense in $L^2(\Omega)$ if and only if its orthogonal in $L^2(\Omega)$ is reduced to zero. As $\{0\} \subset \mathcal{F}^{\perp}(0)$, it remains to show that $\mathcal{F}^{\perp}(0) \subset \{0\}$. Let $\rho^0 \in \mathcal{F}^{\perp}(0)$, then

$$\left\langle \rho^0, z(0) \right\rangle_{L^2(\Omega)} = \int_{\Omega} \rho^0 z(0)\, dx = 0. \tag{49}$$

Where z is solution of (47). It is therefore natural to define the adjoint ρ of z, this is the solution of the following problem

$$\left\{\begin{array}{rcll} -\rho' - \Delta \rho + f'(y_0)\, \rho & = & 0 & \text{in } \mathcal{Q}, \\ \rho(0) & = & \rho^0 & \text{in } \Omega, \\ \rho & = & 0 & \text{on } \Sigma. \end{array}\right. \tag{50}$$

Where ρ is solution of (50).

Now we multiply the first equation of system (47) by ρ. After integration by parts in $\mathcal{Q}$, it comes

$$\begin{aligned} 0 \;=\; & \int\!\!\int_{\Omega\times(0,T)} \rho\left(-z' - \Delta z + f'(y_0)z\right) dxdt + \int_{\Omega} \rho(T)\, z(T)\, dx \\ & + \int\!\!\int_{\Gamma\times(0,T)} \rho \frac{\partial z}{\partial \nu} \Gamma dt - \int\!\!\int_{\Gamma\times(0,T)} \frac{\partial \rho}{\partial \nu} z d\Gamma dt - \int_{\Omega} \rho^0 z(0)\, dx. \end{aligned} \tag{51}$$

Since z and ρ are solutions of (47) and (50) respectively, (51) becomes

$$\int\!\!\int_{\Omega\times(0,T)} \rho u \chi_{\mathcal{O}} dxdt = 0. \tag{52}$$

Therefore, ρ satisfies (50) and (52) and by applying **Lemma 4.1**, we deduce that

$$\rho = 0 \quad \text{in} \quad \mathcal{Q}.$$

As a consequence, $\rho^0 = 0$ which shows that $\mathcal{F}^{\perp}(0) = \{0\}$. □

4.3. Characterization of optimal control

In this section, we will characterize the optimal control using a result of **Fenchel-Rockafellar** duality. The optimality system satisfied by $(\widehat{u}, \widehat{q})$ is established. Let $\rho^0 \in L^2(\Omega)$ and ρ the associated solution of

$$\begin{cases} \rho' - \Delta\rho + f'(y_0)\,\rho &= 0 \quad \text{in } \mathcal{Q}, \\ \rho(0) &= \rho^0 \quad \text{in } \Omega, \\ \rho &= 0 \quad \text{on } \Sigma. \end{cases} \tag{53}$$

We now introduce the functional J_ϵ defined by

$$J_\epsilon\left(\rho^0\right) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\rho|^2 \, dxdt + \epsilon \left\|\rho^0\right\|_{L^2(\Omega)} + \int_0^T \int_{\mathcal{O}} \rho h dxdt. \tag{54}$$

Consider the following unconstrained problem

$$(P_\epsilon) : \begin{cases} \min J_\epsilon\left(\rho^0\right), \\ \rho^0 \in L^2(\Omega). \end{cases} \tag{55}$$

Then, we have

Proposition 4.1.
The functional J_ϵ defined in (54) is coercive.

Proof. To prove that J_ϵ is coercive, it suffices to show the following relation :

$$\lim_{\left\|\rho^0\right\|_{L^2(\Omega)} \to \infty} \frac{J_\epsilon\left(\rho^0\right)}{\left\|\rho^0\right\|_{L^2(\Omega)}} \geq \epsilon. \tag{56}$$

Let $\left(\rho_j^0\right) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system (53) with $\left\|\rho_j^0\right\|_{L^2(\Omega)} \longrightarrow \infty$. We normalize them as follows

$$\widetilde{\rho}_j^0 = \frac{\rho_j^0}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}. \tag{57}$$

So $\left\|\widetilde{\rho}_j^0\right\|_{L^2(\Omega)} \leq 1$. On the other hand, let $\widetilde{\rho}_j$ be the solution of (53) with initial data $\widetilde{\rho}_j^0$. Then, we have

$$\begin{aligned} \frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} &= \frac{1}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \int_0^T \int_{\mathcal{O}} \left(\frac{1}{2}\left|\rho_j\right|^2 + \rho_j h\right) dxdt + \epsilon \\ &= \int_0^T \int_{\mathcal{O}} \widetilde{\rho}_j \left(\frac{1}{2}\rho_j + h\right) dxdt + \epsilon. \end{aligned} \tag{58}$$

We now show that the last integral in equation (58) is bounded. Indeed, we know that ρ_j is the solution of the problem

$$\left\{\begin{array}{rcll} \rho_j' - \Delta\rho_j + f'(y_0)\,\rho_j &=& 0 & \text{in} \quad \mathcal{Q}, \\ \rho_j &=& 0 & \text{on} \quad \Sigma, \\ \rho_j(0) &=& \rho_j^0 & \text{in} \quad \Omega. \end{array}\right. \tag{59}$$

Multiplying the first equation of system (59) by ρ_j then **integrating by parts** on $\mathcal{Q}$, yields

$$\begin{aligned} 0 \;=\; & \int_0^T \int_\Omega \left(\rho_j' - \Delta\rho_j + f'(y_0)\,\rho_j\right)\rho_j\,dxdt = \frac{1}{2}\left\|\rho_j(T)\right\|^2_{L^2(\Omega)} \\ & -\frac{1}{2}\left\|\rho_j^0\right\|^2_{L^2(\Omega)} + \left\|\nabla\rho_j\right\|^2_{L^2(\mathcal{Q})}. \end{aligned} \tag{60}$$

By the **Poincaré inequality**, (60) becomes,

$$C_0\left\|\rho_j\right\|^2_{L^2(\mathcal{Q})} \leq \left\|\nabla\rho_j\right\|^2_{L^2(\mathcal{Q})} \leq \frac{1}{2}\left\|\rho_j^0\right\|^2_{L^2(\Omega)}. \tag{61}$$

Now, by **Cauchy Schwartz inequality**, one finds

$$\int_0^T \int_{\mathcal{O}} \frac{h\rho}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}dxdt \leq C_1 \frac{\left\|\rho_j\right\|_{L^2(\mathcal{Q})}}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}. \tag{62}$$

From (61), (62), we conclude that

$$\int_0^T \int_{\mathcal{O}} \frac{h\rho}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}dxdt \leq C. \tag{63}$$

Returning to relation (58), two cases can occur :

1. $\int_0^T \int_{\mathcal{O}} \widetilde{\rho}_j^2\,dxdt > 0$. In this case, we immediately obtain

$$\frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \underset{\left\|\rho_j^0\right\|_{L^2(\Omega)} \longmapsto +\infty}{\longrightarrow} +\infty. \tag{64}$$

2. $\int_0^T \int_{\mathcal{O}} \widetilde{\rho}_j^2\,dxdt = 0$.In this case, since $\left(\widetilde{\rho}_j^0\right)_j$ is bounded in $L^2(\Omega)$, we can extract a subsequence $\left(\widetilde{\rho}_j^0\right)_j$ such that :

$$\left\{\begin{array}{l} \widetilde{\rho}_j^0 \rightharpoonup \psi^0 \text{ weakly in } L^2(\Omega), \\ \widetilde{\rho}_j \rightharpoonup \psi \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right). \end{array}\right. \tag{65}$$

Where ψ is solution of system (53) with initial data ψ^0. Moreover, by lower semi continuity of the norm, it comes

$$\int_0^T \int_{\mathcal{O}} |\psi|^2\,dxdt \leq \liminf \int_0^T \int_{\mathcal{O}} \left|\widetilde{\rho}_j\right|^2 dxdt = 0. \tag{66}$$

Therefore,

$$\psi = 0 \quad \text{in} \quad \mathcal{O}\times(0,T). \tag{67}$$

And as ψ is solution of (53), and in view of (67), we have

$$\psi = 0 \quad \text{in} \quad \Omega \times (0, T). \tag{68}$$

Thus,

$$\widetilde{\rho}_j \rightharpoonup 0 \text{ weakly in } L^2\left(0, T; H_0^1(\Omega)\right). \tag{69}$$

Moreover, from inequality (61), we deduce that $\left(\frac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}\right)_j$ is bounded in $L^2\left(0, T; H_0^1(\Omega)\right)$. Hence

$$\frac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \rightharpoonup \xi \text{ in } L^2\left(0, T; H_0^1(\Omega)\right). \tag{70}$$

But,

$$\widetilde{\rho}_j = \frac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \rightharpoonup 0. \tag{71}$$

From (70) and (71), we conclude that

$$\xi' - \Delta\xi + f'(y_0)\,\xi = 0 \quad \text{in} \quad L^2(\mathcal{Q}). \tag{72}$$

So by **Lemma 4.1**, it comes

$$\xi = 0 \quad \text{in} \quad \mathcal{Q}. \tag{73}$$

As a consequence,

$$\widetilde{\rho}_j = \frac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \longmapsto 0. \tag{74}$$

But,

$$\frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} = \frac{1}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \int_0^T \int_{\mathcal{O}} \left(\frac{1}{2}\left|\rho_j\right|^2 + \rho_j h\right) dxdt + \epsilon. \tag{75}$$

Thus,

$$\liminf_{j \longmapsto +\infty} \frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \geq \epsilon. \tag{76}$$

Hence relation (56) is satisfied. □

Theorem 4.2.
Problem (55) has a unique solution $\widehat{\rho}^0 \in L^2(\Omega)$. Furthermore, if $\widehat{\rho}$ is the solution of (53) associated to $\widehat{\rho}^0$, then $(\widehat{u} = \widehat{\rho}, q)$ is solution such that (58), (60) and (56) hold.

Proof. As J_ϵ attains its minimum value at $\widehat{\rho}^0 \in L^2(\Omega)$, then, for any $\psi^0 \in L^2(\Omega)$ and any $r \in \mathbb{R}$ we have

$$J_\epsilon\left(\widehat{\rho}^0\right) \leq J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) \Longrightarrow J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) - J_\epsilon\left(\widehat{\rho}^0\right) \geq 0. \tag{77}$$

On the other hand,

$$J_\epsilon\left(\widehat{\rho}^0\right)=\int_0^T\int_{\mathcal{O}}\left(\frac{1}{2}\left|\widehat{\rho}\right|^2+\widehat{\rho}h\right)dxdt+\epsilon\left\|\widehat{\rho}^0\right\|_{L^2(\Omega)}.$$

$$\begin{aligned}J_\epsilon\left(\widehat{\rho}^0+r\psi^0\right) &= \frac{1}{2}\int_0^T\int_{\mathcal{O}}\left|\widehat{\rho}\right|^2dxdt+\frac{r^2}{2}\int_0^T\int_{\mathcal{O}}\left|\psi\right|^2dxdt \\ &+r\int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi dxdt+\sqrt{\varepsilon}\left\|\widehat{\rho}^0+r\psi^0\right\|_{L^2(\Omega)} \\ &+\int_0^T\int_{\mathcal{O}}h\left(\widehat{\rho}+r\psi\right)dxdt.\end{aligned} \tag{78}$$

Substituting (78) in (77) and after simplifications, we find

$$\begin{aligned}0 &\leq J_\epsilon\left(\widehat{\rho}^0+r\psi^0\right)-J_\varepsilon\left(\widehat{\rho}^0\right) \\ 0 &\leq \frac{r^2}{2}\int_0^T\int_{\mathcal{O}}\left|\psi\right|^2dxdt+\epsilon\left[\left\|\widehat{\rho}^0+r\psi^0\right\|_{L^2(\Omega)}-\left\|\widehat{\rho}^0\right\|_{L^2(\Omega)}\right] \\ &+r\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt.\end{aligned} \tag{79}$$

On the other hand,

$$\left\|\widehat{\rho}^0+r\psi^0\right\|_{L^2(\Omega)}-\left\|\widehat{\rho}^0\right\|_{L^2(\Omega)}\leq\left|r\right|.\left\|\psi^0\right\|_{L^2(\Omega)}. \tag{80}$$

From (79) and (80), we obtain for any $\psi^0\in L^2\left(\Omega\right)$ and $r\in\mathbb{R}$,

$$0\leq\frac{r^2}{2}\int_0^T\int_{\mathcal{O}}\left|\psi\right|^2dxdt+\epsilon\left|r\right|.\left\|\psi^0\right\|_{L^2(\Omega)}+r\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt.$$

Dividing by $r>0$ and by passing to the limit $r\to 0$, we obtain

$$\epsilon.\left\|\psi^0\right\|_{L^2(\Omega)}+\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt\geq 0.$$

The same calculations with $r<0$ give

$$\left|\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt\right|\leq\epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\forall\psi^0\in L^2\left(\Omega\right).$$

so if we take $\widehat{u}=\widehat{\rho}\chi_{\mathcal{O}}$ in (58) and we multiply the first equation of the system (58) by ψ solution of (53) and we get after integration by parts over $\mathcal{Q}$,

$$\int_\Omega q(0)\psi^0dx=\int_0^T\int_{\mathcal{O}}\left(h+\widehat{\rho}\right)\psi dxdt. \tag{81}$$

It comes from the last two relations:

$$\left|\int_\Omega q(0)\psi^0dx\right|\leq\epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\forall\psi^0\in L^2\left(\Omega\right).$$

Consequently,

$$\left\|q\left(x,0\right)\right\|_{L^2(\Omega)}\leq\epsilon. \tag{82}$$

□

4.4. Use of the concept of sentinel : Detection of pollution and furtivity

It is noted that

$$\mathcal{S}(\lambda,\tau) \simeq \mathcal{S}(0,0) + \lambda \frac{\partial \mathcal{S}}{\partial \lambda}(0,0) + \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0). \tag{83}$$

And

$$\text{observation of } y = y\chi_{\mathcal{O}} = \text{function } m_0(x,t) \text{ of } L^2(\mathcal{O}\times(0,T)). \tag{3.2}$$

With the notation (62) for the observation of y, and while using (55), one thus has

$$\lambda \frac{\partial \mathcal{S}}{\partial \lambda}(0,0) \simeq \int\int_{\mathcal{O}\times(0,T)} (h+u)\, m_0 dxdt - \mathcal{S}(0,0) - \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0). \tag{84}$$

Such as

$$\mathcal{S}(0,0) = \int\int_{\mathcal{O}\times(0,T)} (h+u)\, y_0 dxdt.$$

But

$$\lambda \frac{\partial \mathcal{S}}{\partial \lambda}(0,0) = \int\int_{\mathcal{O}\times(0,T)} (h+u)\, y_\lambda dxdt. \tag{85}$$

In (85), y_λ is defined by

$$\begin{cases} y_\lambda' - \Delta y_\lambda + f'(y_0)\, y_\lambda &= \widehat{\xi} \quad \text{in} \quad \mathcal{Q}, \\ y_\lambda(0) &= 0 \quad \text{in} \quad \Omega, \\ y_\lambda &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{86}$$

By multiplying the corresponding equation (29) by y_λ, one finds, after integration by parts, that

$$\frac{\partial \mathcal{S}}{\partial \lambda}(0,0) = \int\int_{\mathcal{O}\times(0,T)} q_0 \widehat{\xi} dxdt. \tag{87}$$

Consequently

$$\int\int_{\mathcal{O}\times(0,T)} (q_0+z)\left\{\lambda\widehat{\xi}\right\} dxdt \simeq \int\int_{\mathcal{O}\times(0,T)} (h+\rho)(m_0-y_0)\, dxdt - \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0). \tag{88}$$

So

$$\int\int_{\mathcal{O}\times(0,T)} q_0 \left\{\lambda\widehat{\xi}\right\} dxdt \simeq \int\int_{\mathcal{O}\times(0,T)} (h+\rho)\, |m_0-y_0|\, dxdt + \tau\epsilon. \tag{89}$$

the quantity (89) which is estimated by the 1st member of (83).

Pollution $\lambda\widehat{\xi}$ is furtive for the sentinel defined by h if

$$\int\int_{\mathcal{O}\times(0,T)} q_0 \left\{\lambda\widehat{\xi}\right\} dxdt = 0. \tag{90}$$

There are thus always furtive pollution for a sentinel.

5. Conclusion

In this work we present the weak and pointwise sentinel to estimate the pollution term in diffusion equation when the state governed by unknown datum and missing initial condition when the classical approach of sentinel method gives us information related to the missing data for this we try to avoided this problems by notion of control. This method can be also used in pointwise sentinel and weakly sentinel.

Funding

This work was supported by the Directorate-General for Scientific Research and Technological Development (DGRSDT).

Competing Interests

All authors have no conflict of interest to report.

Acknowledgements

The authors thank the referees for their careful reading and their precious comments. Their help is much appreciated.

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