Innovative Journal of Mathematics
v. 1 no. 2, 71-81, (2022)
doi:10.55059/ijm.2022.1.2/27

ISSN 2790-3370 (print)
ISSN 2790-3389 (online)

Generalized order n fractional integrals

Research Article

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Abstract: We derive a type of fractional integral with parameters in the integrand for an arbitrary amount of integrals. Therefore we solve a wide class of integrals which can be respresented for various n.

MSC: 33E99, 11Y35

Keywords: Integration • Fractional part • Recursive sequence•

Received 2022-02-06; Accepted 2022-03-08; Published 2022-04-14

1. Introduction

Integrals have been an ongoing topic in the mathematical analysis since they have been discovered. Integrals are widely used in evaluation of various series, see [2],[5],[9],[11]. Many books have been written about them, see [3],[10]. The topic we will discuss today are integrals with fractional part. Some of them can be found here [4],[8]. In this paper we give a generalization of the integral of the fractional part, both in terms of the arbitrary power which occurs in the integrand and in terms of the amout of integrals, many integrals involving fractional part can be found here [6]. The recursive sequence is found that links every integral of the sequence with the integral defined as C_n in the introductory.

We give our first important and well known definition

Definition 1.1.

The function $\{\}$ denotes the fractional part of a function, the function $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. They are related to the function variable in the following relation

$$\{x\} = x - \lfloor x \rfloor.$$

More about the usage of the fractional and integer part as a function, see [7].

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Let us give a few examples of the fractional part function.

$a)$ $\{\pi\} = 0.14....$

$b)$ $\{2.25\} = 0.25$

Now a few examples of the integer part.

$a)$ $\lfloor 4.5 \rfloor = 4$

$b)$ $\lfloor \pi \rfloor = 3$

Let us define the integral sequences we will discuss.

Definition 1.2.
The sequence C_n is defined in the following way

$$C_n = \underset{V_{n-1}[0,1]}{\int \cdots \int} \int_{\frac{1}{x_2 \cdots x_n}}^{+\infty} \left\{ \frac{q}{y^k} \right\} y^{m-2} \frac{dy dx_n ... dx_2}{x_2 x_3 ... x_n}.$$

The sequence I_n is given by

$$I_n = \underset{V_n[0,1]}{\int \cdots \int} \left\{ q \left(\frac{x_n x_{n-1} ... x_2}{x_1} \right)^k \right\} \left(\frac{x_1}{x_n x_{n-1} ... x_2} \right)^m dx_n dx_{n-1} .. dx_1.$$

The next definition will be extensively used in solving the single and double fractional part integrals.

Definition 1.3.
The Hurwitz zeta function, see [1] is defined for $\Re(s) > 1$ and $a \neq 0, -1, -2..$ by

$$\zeta(s, a) = \sum_{n=0}^{+\infty} \frac{1}{(n+a)^s}$$

The next definition is used to give a closed form of the C_n integral.

Definition 1.4.
The upper incomplete gamma function, see [1] is defined as

$$\Gamma(s, x) = \int_x^{+\infty} t^{s-1} e^{-t} dt$$

where s is a complex parameter, whose real part is positive.

In the next section we begin with the simplest form of the integral we will discuss.

2. Main results

First Theorem of our paper plays a role in evaluating the I_2 integral.

Theorem 2.1.
The following equality holds for $k \in \left(0, \frac{1}{2}\right), q \in \mathbb{N}$

$$\int_0^1 \left\{\frac{q}{x^k}\right\} dx$$

$$= \left(\frac{k}{k-1}\zeta\left(-1+\frac{1}{k}, 1+q\right) + k\zeta\left(-1+\frac{1}{k}, 1+q\right) - k\zeta\left(\frac{1}{k}, q+1\right)\right)\frac{q^{\frac{1}{k}}}{k}$$

$$+\left(-\frac{k}{k-1}\zeta\left(-1+\frac{1}{k}, q\right) - k\zeta\left(-1+\frac{1}{k}, q\right)\right)\frac{q^{\frac{1}{k}}}{k}.$$

Proof. Let us observe the integral

$$\int_0^1 \left\{\frac{q}{x^k}\right\} dx.$$

We introduce a substitution $\frac{q}{x^k} = t$ which gives us

$$\int_0^1 \left\{\frac{q}{x^k}\right\} dx = \frac{q^{\frac{1}{k}}}{k}\int_q^{+\infty} \frac{\{t\}}{t^{1+\frac{1}{k}}} dt$$

Let us call a constant $\frac{q^{\frac{1}{k}}}{k}$ c to minimize the clutter in the formulas. We need to get rid of the fractional part, therefore we write it in terms of the sum and the integral as follows

$$\int_0^1 \left\{\frac{q}{x^k}\right\} dx = c\sum_{m=q}^{+\infty}\int_m^{m+1} \frac{t-m}{t^{1+\frac{1}{k}}} dt$$

We will focus ourselves onto the integral evaluation and then sum each expression we get.

$$\int_m^{m+1} \frac{t-m}{t^{1+\frac{1}{k}}} dt = \left(\frac{k(t+(k-1)m)}{(k-1)t^{\frac{1}{k}}}\right)\Bigg|_m^{m+1}$$

$$= \frac{k(m+1+(k-1)m)}{(k-1)(m+1)^{\frac{1}{k}}} - \frac{k\,(m+(k-1)m)}{(k-1)m^{\frac{1}{k}}}$$

$$= \frac{k(m+1)}{(k-1)(m+1)^{\frac{1}{k}}} + \frac{mk}{(m+1)^{\frac{1}{k}}} - \frac{km^{1-\frac{1}{k}}}{(k-1)} - km^{1-\frac{1}{k}}$$

Summing each of the expressions and multiplying all of them with the constant $c = \frac{q^{\frac{1}{k}}}{k}$ we get that

$$\int_0^1 \left\{\frac{q}{x^k}\right\} dx = \left(\frac{k}{k-1}\zeta\left(-1+\frac{1}{k}, 1+q\right) + k\zeta\left(-1+\frac{1}{k}, 1+q\right) - k\zeta\left(\frac{1}{k}, q+1\right)\right)\frac{q^{\frac{1}{k}}}{k}$$

$$+\left(-\frac{k}{k-1}\zeta\left(-1+\frac{1}{k}, q\right) - k\zeta\left(-1+\frac{1}{k}, q\right)\right)\frac{q^{\frac{1}{k}}}{k}.$$

The proof is done. □

We give our first corollary.

Corollary 2.1.
Setting $k=\frac{1}{3}$ and $q=2$ in Theorem 2.1 we get

$$\int_0^1 \left\{\frac{2}{x^{\frac{1}{3}}}\right\}dx$$

$$= 24\left(-\frac{\zeta(3,3)}{3}+\frac{1}{2}\left(\frac{\pi^2}{6}-1\right)+\frac{1}{3}\left(\frac{\pi^2}{6}-\frac{5}{4}\right)+\frac{1}{3}\left(1-\frac{\pi^2}{6}\right)+\frac{1}{2}\left(\frac{5}{4}-\frac{\pi^2}{6}\right)\right)$$
$$= 1-8\zeta(3,3).$$

The following Theorem will be used in evaluating the I_2 integral.

Theorem 2.2.
The following equality holds for
1. $q\in\mathbb{N}, k>0, m<1$
2. $q\in\mathbb{N}, k<0, m>1$

$$\int_1^\infty \left\{\frac{q}{x^k}\right\}x^{m-2}dx$$

$$= \left(\sum_{l=1}^{q} -\frac{k\left((m-1)l-(l-1)m+(k+1)l-k-1\right)}{(m-1)(m-k-1)l^{\frac{m-1}{k}}}\right.$$
$$\left.+\sum_{l=1}^{q}\frac{k\left((m-1)(l-1)-(l-1)m+(k+1)l-k-1\right)}{(m-1)(m-k-1)(l-1)^{\frac{m-1}{k}}}\right)\cdot\frac{q^{\frac{m-1}{k}}}{k}.$$

Proof. Observing the integral

$$\int_1^\infty \left\{\frac{q}{x^k}\right\}x^{m-2}dx.$$

Introducing a substitution $\frac{q}{x^k}=y$ we get

$$\int_1^\infty \left\{\frac{q}{x^k}\right\}x^{m-2}dx = \frac{q^{\frac{m-1}{k}}}{k}\int_0^q \{y\}\frac{1}{y^{\frac{k+m-1}{k}}}dy$$

Calling $\frac{q^{\frac{m-1}{k}}}{k}$ a constant c to minimize the clutter. Writing it as a sum and taking a fractional part, we get

$$\int_1^\infty \left\{\frac{q}{x^k}\right\}x^{m-2}dx = c\sum_{l=1}^{q}\int_{l-1}^{l}\frac{y-l+1}{y^{\frac{k+m-1}{k}}}dy$$

Focusing onto the integral, we get

$$\int_{l-1}^{l}\frac{y-l+1}{y^{\frac{k+m-1}{k}}}dy = -\left(\frac{k((m-1)y-(l-1)m+(k+1)l-k-1)}{(m-1)(m-k-1)y^{\frac{m-1}{k}}}\right)\Bigg|_{l-1}^{l}$$

$$= -\frac{k((m-1)l-(l-1)m+(k+1)l-k-1)}{(m-1)(m-k-1)l^{\frac{m-1}{k}}}$$
$$+\frac{k((m-1)(l-1)-(l-1)m+(k+1)l-k-1)}{(m-1)(m-k-1)(l-1)^{\frac{m-1}{k}}}$$

Now summing each of the terms we get the result

$$\int_1^\infty \left\{\frac{q}{x^k}\right\} x^{m-2}dx = \left(\sum_{l=1}^{q} -\frac{k\left((m-1)l-(l-1)m+(k+1)l-k-1\right)}{(m-1)(m-k-1)l^{\frac{m-1}{k}}}\right.$$

$$\left. + \sum_{l=1}^{q} \frac{k\left((m-1)(l-1)-(l-1)m+(k+1)l-k-1\right)}{(m-1)(m-k-1)(l-1)^{\frac{m-1}{k}}}\right) \cdot \frac{q^{\frac{m-1}{k}}}{k}$$

The proof is done. □

Corollary 2.2.
Setting $m=\frac{1}{2}, q=2, k=6$ in the last Theorem, we get

$$\int_1^\infty \left\{\frac{2}{x^6}\right\} x^{\frac{1}{2}-2}dx = \frac{12-\frac{132\sqrt[12]{2}}{13}}{6\sqrt[12]{2}} = 0.195441$$

The following Theorem is part of the I_2 integral.

Theorem 2.3.
The following equality holds for
1. $k=\frac{1}{2}, m>0$
2. $k>\frac{1}{2}, 0<m<\frac{k}{2k-1}$
3. $0<k<\frac{1}{2}, m>0$
4. $0<k<\frac{1}{2}, m<\frac{k}{2k-1}$
5. $0<k, 0<m<\frac{k}{2k-1}$

$$\int_0^1 \left\{\frac{q}{x^k}\right\} x^m dx$$

$$= \left(\frac{km}{((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m}, 1+q\right)\right.$$

$$+\frac{km(k-1)m}{(m+k)((k-1)m-k)}\left(\zeta\left(\frac{1}{k}+\frac{1}{m}-1, q+1\right)-\zeta\left(\frac{1}{m}+\frac{1}{k}, q+1\right)\right)$$

$$-\frac{k^2m}{(m+k)((k-1)m-k)}\left(\zeta\left(\frac{1}{k}+\frac{1}{m}-1, q+1\right)-\zeta\left(\frac{1}{m}+\frac{1}{k}, q+1\right)\right)$$

$$-\frac{km}{((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m}, q\right)-\frac{k(k-1)m^2}{(m+k)((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m}, q\right)$$

$$\left.+\frac{k^2m}{(m+k)((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m}, q\right)\right)\cdot\frac{q^{\frac{1}{k}+\frac{1}{m}}}{k}$$

Proof. The proof is similar to the proof we gave in Theorem 2.1 therefore it is omitted. □

The following Theorem represents the I_2 integral.

Theorem 2.4.
The following equality holds for
1.$k \in (0,1], m \in (0,1)$
2.$k \in (1,+\infty), m \in (0, \frac{k}{2k-1})$
3.$k \in (0,\frac{1}{2}), m \in (0, \frac{k}{2k-1})$

$$\int_0^1 \int_0^1 \left\{ q \left(\frac{y}{x}\right)^k \right\} \left(\frac{x}{y}\right)^m dydx$$

$$= \frac{1}{2}\Bigg(\Bigg(\sum_{l=1}^{q} -\frac{k\left((m-1)l-(l-1)m+(k+1)l-k-1\right)}{(m-1)(m-k-1)l^{\frac{m-1}{k}}}$$

$$+\sum_{l=1}^{q} \frac{k\left((m-1)(l-1)-(l-1)m+(k+1)l-k-1\right)}{(m-1)(m-k-1)(l-1)^{\frac{m-1}{k}}}\Bigg) \cdot \frac{q^{\frac{m-1}{k}}}{k}\Bigg)$$

$$+\frac{1}{2}\Bigg(\Bigg(\frac{km}{((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m},1+q\right)$$

$$+\frac{km(k-1)m}{(m+k)((k-1)m-k)}\left(\zeta\left(\frac{1}{k}+\frac{1}{m}-1,q+1\right)-\zeta\left(\frac{1}{m}+\frac{1}{k},q+1\right)\right)$$

$$-\frac{k^2m}{(m+k)((k-1)m-k)}\left(\zeta\left(\frac{1}{k}+\frac{1}{m}-1,q+1\right)-\zeta\left(\frac{1}{m}+\frac{1}{k},q+1\right)\right)$$

$$-\frac{km}{((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m},q\right)-\frac{k(k-1)m^2}{(m+k)((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m},q\right)$$

$$+\frac{k^2m}{(m+k)((k-1)m-k)}\zeta\left(-1+\frac{1}{k}+\frac{1}{m},q\right)\Bigg)\cdot\frac{q^{\frac{1}{k}+\frac{1}{m}}}{k}\Bigg)$$

Proof. Observing the integral

$$\int_0^1 \int_0^1 \left\{ q \left(\frac{y}{x}\right)^k \right\} \left(\frac{x}{y}\right)^m dydx.$$

We make a substitution $\frac{x}{y}=t$ from which we get

$$\int_0^1 \int_0^1 \left\{ q \left(\frac{y}{x}\right)^k \right\} \left(\frac{x}{y}\right)^m dydx = \int_0^1 x \int_x^{+\infty} \left\{\frac{q}{t^k}\right\} t^m \frac{dt}{t^2} dx$$

We will perform partial integration taking

$$f(x) = \int_x^{+\infty} \left\{\frac{q}{t^k}\right\} t^m \frac{dt}{t^2}, f^{'}(x) = -\left\{\frac{q}{x^k}\right\} x^{m-2}, g(x)=x, \int g(x) = \frac{x^2}{2}$$

we get

$$\int_0^1 \int_0^1 \left\{ q \left(\frac{y}{x}\right)^k \right\} \left(\frac{x}{y}\right)^m dydx = \left(\int_x^{+\infty} \left\{\frac{q}{t^k}\right\} t^m \frac{dt}{t^2}\cdot\frac{x^2}{2}\right)\Bigg|_0^1 + \frac{1}{2}\int_0^1 \left\{\frac{q}{x^k}\right\} x^m dx$$

$$= \frac{1}{2}\int_1^{+\infty}\left\{\frac{q}{t^k}\right\} t^{m-2} dt + \frac{1}{2}\int_0^1 \left\{\frac{q}{x^k}\right\} x^m dx$$

Substituting Theorems 2.2 and 2.3 the result follows. The proof is done. □

Corollary of the previously derived Theorem is given.

Corollary 2.3.
Setting $q = 2, k = \frac{1}{3}, m = \frac{1}{6}$ *we get that*

$$\int_0^1 \int_0^1 \left\{ 2 \left(\frac{y}{x} \right)^{\frac{1}{3}} \right\} \left(\frac{x}{y} \right)^{\frac{1}{6}} dydx = 0.587629 \sim \frac{77 \ln \pi}{150}$$

The following Theorem gives a recursive sequence used to calculate I_n for arbitrary n.

Theorem 2.5.
The following relation holds for $n \geq 3$ *and in the following cases*
1.$k \in (0,1], m \in (0,1)$
2.$k \in (1,+\infty), m \in (0, \frac{k}{2k-1})$
3.$k \in (0, \frac{1}{2}), m \in (0, \frac{k}{2k-1})$

$$I_n = \underset{V_n[0,1]}{\int \cdots \int} \left\{ q \left(\frac{x_n x_{n-1} \dots x_2}{x_1} \right)^k \right\} \left(\frac{x_1}{x_n x_{n-1} \dots x_2} \right)^m dx_n dx_{n-1} .. dx_1$$

$$= \frac{1}{2} I_{n-1} + \frac{1}{2} C_{n-1}.$$

Proof. Let us consider the integral

$$I_n = \underset{V_n[0,1]}{\int \cdots \int} \left\{ q \left(\frac{x_n x_{n-1} \dots x_2}{x_1} \right)^k \right\} \left(\frac{x_1}{x_n x_{n-1} \dots x_2} \right)^m dx_n dx_{n-1} .. dx_1$$

We will introduce a substitution $\frac{x_1}{x_n \dots x_2} = y$ which gives us

$$I_n = \int_0^1 x_1 \underset{V_{n-2}[0,1]}{\int \cdots \int} \int_{\frac{x_1}{x_2 \cdots x_{n-1}}}^{+\infty} \left\{ \frac{q}{y^k} \right\} y^{m-2} \frac{dy dx_{n-1} \dots dx_1}{x_2 \dots x_{n-1}}$$

Performing partial integration while taking

$$f(x_1) = \underset{V_{n-2}[0,1]}{\int \cdots \int} \int_{\frac{x_1}{x_2 \cdots x_{n-1}}}^{+\infty} \left\{ \frac{q}{y^k} \right\} y^{m-2} \frac{dy dx_{n-1} \dots dx_2}{x_2 \dots x_{n-1}}, dv = x_1 dx_1$$

we get that

$$I_n = \left(\frac{x_1^2}{2} \underset{V_{n-2}[0,1]}{\int \cdots \int} \int_{\frac{x_1}{x_2 \cdots x_{n-1}}}^{+\infty} \left\{ \frac{q}{y^k} \right\} y^{m-2} \frac{dy dx_{n-1} \dots dx_2}{x_2 \dots x_{n-1}} \right) \Bigg|_0^1$$

$$+ \frac{1}{2} \underset{V_{n-1}[0,1]}{\int \cdots \int} \left\{ q \left(\frac{x_{n-1} x_{n-2} \dots x_2}{x_1} \right)^k \right\} \left(\frac{x_1}{x_{n-1} x_{n-2} \dots x_2} \right)^m dx_{n-1} dx_{n-2} .. dx_1$$

$$= \frac{1}{2} C_{n-1} + \frac{1}{2} I_{n-1}$$

□

The following Theorem simplifies the evaluation of the C_n integral and makes the recursive relation much more useful.

Theorem 2.6.
Let C_n denote the following integral

$$C_n = \underset{V_{n-1}[0,1]}{\int \cdots \int} \int_{\frac{1}{x_2 \cdots x_n}}^{+\infty} \left\{ \frac{q}{y^k} \right\} y^{m-2} \frac{dy dx_n ... dx_2}{x_2 x_3 ... x_n}.$$

Then the following equality holds.

$$C_n = \int_0^1 \frac{\ln^{n-1}(x_2)}{(n-1)!} \cdot (-1)^{n-1} \left(\frac{1}{x_2} \right)^m \left\{ q x_2^k \right\} dx_2$$

Proof. We will need the following Lemma.

Lemma 2.1.
The following equality holds

$$C_n = \int_0^1 \frac{\ln^l(x_2)}{l!} (-1)^l \underset{V_{n-l-1}[0,1]}{\int \cdots \int} \left(\frac{1}{x_2 ... x_{n-l+1}} \right)^m \left\{ q (x_2 ... x_{n-l+1})^k \right\} dx_{n-l+1} ... dx_2.$$

Proof. We will prove the Lemma by induction on l.
Base: for $l = 1$ we have

$$C_n = \int_0^1 \frac{1}{x_2} ... \int_0^1 \frac{1}{x_n} \int_{\frac{1}{x_2 \cdots x_n}}^{+\infty} \left\{ \frac{q}{y^k} \right\} y^{m-2} dy dx_n ... dx_2$$

We will prove it using partial integration.
Taking $f(x_2) = \int_0^1 .... \int_0^1 \frac{1}{x_n} \int_{\frac{1}{x_2 .... x_n}}^{+\infty} \left\{ \frac{q}{y^k} \right\} y^{m-2} dy dx_n ... dx_3$ and $dv = \frac{dx_2}{x_2}$.
From partial integration we get

$$C_n = -\int_0^1 \ln(x_2) \underset{V_{n-2}[0,1]}{\int \cdots \int} \left(\frac{1}{x_2 ... x_n} \right)^m \left\{ q (x_2 ... x_n)^k \right\} dx_n ... dx_2$$

which is true since putting $l = 1$ in the Lemma the expression follows.
Induction hypothesis: Let us assume that the formula is valid for $1 \leq l \leq n-1$ then from the induction hypothesis the formula is valid for $l+1$.
Inductive step: Using a substitution $p = \frac{1}{(x_2 \cdots x_{n-l+1})}$ on the hypothesis

$$C_n = \int_0^1 \frac{\ln^l(x_2)}{l!} (-1)^l \underset{V_{n-l-1}[0,1]}{\int \cdots \int} \left(\frac{1}{x_2 ... x_{n-l+1}} \right)^m \left\{ q (x_2 ... x_{n-l+1})^k \right\} dx_{n-l+1} ... dx_2$$

we get

$$C_n = \int_0^1 \frac{\ln^l(x_2)}{x_2 l!} (-1)^l \underset{V_{n-2-l}[0,1]}{\int \cdots \int} \int_{\frac{1}{x_2 \cdots x_{n-l}}}^{+\infty} p^{m-2} \left\{ \frac{q}{p^k} \right\} \frac{dp dx_{n-l} ... dx_2}{x_3 ... x_{n-l}}$$

Doing partial integration while taking

$$f(x_2) = \int_{V_{n-2-l}[0,1]} \cdots \int \int_{\frac{1}{x_2 \dots x_{n-l}}}^{+\infty} p^{m-2} \left\{ \frac{q}{p^k} \right\} \frac{dp dx_{n-l} \dots dx_3}{x_3 \dots x_{n-l}}$$

$$dv = \frac{\ln^l(x_2) dx_2}{x_2} \cdot \frac{(-1)^l}{l!}$$

we get that

$$C_n = \int_0^1 \frac{\ln^{l+1}(x_2)}{(l+1)!} \cdot (-1)^{l+1} \int_{V_{n-(l+1)-1}[0,1]} \cdots \int \left(\frac{1}{(x_2 \dots x_{n-l})} \right)^m \left\{ q(x_2 \dots x_{n-l})^k \right\} dx_{n-l} \dots dx_2$$

Which is the induction hypothesis for $l+1$ and therefore the formula is true. □

Taking $l = n-1$ in the Lemma

$$C_n = \int_0^1 \frac{\ln^l(x_2)}{l!} (-1)^l \int_{V_{n-l-1}[0,1]} \cdots \int \left(\frac{1}{x_2 \dots x_{n-l+1}} \right)^m \left\{ q(x_2 \dots x_{n-l+1})^k \right\} dx_{n-l+1} \dots dx_2.$$

We get the statement of the Theorem

$$C_n = \int_0^1 \frac{\ln^{n-1}(x_2)}{(n-1)!} \cdot (-1)^{n-1} \left(\frac{1}{x_2} \right)^m \left\{ q x_2^k \right\} dx_2.$$

The theorem is proved. □

We proceed to give a simplified expression for the C_n.

Theorem 2.7.
The following equality holds

$$C_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{q^{\frac{m}{k} - \frac{1}{k}}}{k^n} \sum_{l=1}^{q} \int_{l-1}^{l} \frac{(\ln s - \ln q)^{n-1} (s - l + 1)}{s^{\frac{m+k-1}{k}}} ds$$

Proof. We begin with the form

$$C_n = \int_0^1 \frac{\ln^{n-1}(x_2)}{(n-1)!} \cdot (-1)^{n-1} \left(\frac{1}{x_2} \right)^m \left\{ q x_2^k \right\} dx_2.$$

Setting $q x_2^k = s$ we get

$$C_n = \frac{(-1)^{n-1}}{(n-1)!} \int_0^q \left(\ln \left(\left(\frac{s}{q} \right)^{\frac{1}{k}} \right) \right)^{n-1} \left(\left(\frac{q}{s} \right)^{\frac{1}{k}} \right)^m \frac{\{s\}}{k s^{1-\frac{1}{k}} q^{\frac{1}{k}}} ds$$

Using the same idea as in Theorem 2.2 of rewriting the integral as a sum of integrals we get

$$C_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{q^{\frac{m}{k} - \frac{1}{k}}}{k^n} \sum_{l=1}^{q} \int_{l-1}^{l} \frac{(\ln s - \ln q)^{n-1} (s - l + 1)}{s^{\frac{m+k-1}{k}}} ds.$$

For arbitrary $n \in \mathbb{N}$ the integral in question evaluates at

$$\int \frac{(s-l+1)(\ln s)^n}{s^{\frac{m+k-1}{k}}} ds$$

$$= k(\ln s)^n \left(\frac{(l-1)\left(\frac{(m-1)\log(s)}{k}\right)^{-n} \Gamma\left(n+1, \frac{(m-1)\log(s)}{k}\right)}{m-1} \right)$$

$$+k(\ln s)^n \left(\frac{\left(\frac{(-k+m-1)\log(s)}{k}\right)^{-n} \Gamma\left(n+1, -\frac{(k-m+1)\log(s)}{k}\right)}{k-m+1} \right)$$

where $\Gamma(a,x)$ is the incomplete gamma function.
The theorem is proved. □

The following Corollary shows the usage of the recursive relation.

Corollary 2.4.
Let us employ the Theorems we have derived. From Theorem 2.5 we have the following relation and conditions
1. $k \in (0,1], m \in (0,1)$
2. $k \in (1,+\infty), m \in (0, \frac{k}{2k-1})$
3. $k \in (0, \frac{1}{2}), m \in (0, \frac{k}{2k-1})$

$$I_n = \frac{1}{2} I_{n-1} + \frac{1}{2} C_{n-1}.$$

Let us set $k = \phi\pi e, m = \frac{2\phi\pi e}{4\phi\pi e-1}, q = 2, n = 3.$*, where* e *is Eulers constant,* ϕ *golden ratio. From the relation in Theorem 2.5 we get*

$$I_3 = \frac{1}{2} I_2 + \frac{1}{2} C_2$$

We recall the Thereom 2.4 for the I_2 *integral , setting values there. For* C_2 *integral we refer to the Theorem 2.7, setting values there*

$$C_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{q^{\frac{m}{k}-\frac{1}{k}}}{k^n} \sum_{l=1}^{q} \int_{l-1}^{l} \frac{(\ln s - \ln q)^{n-1}(s-l+1)}{s^{\frac{m+k-1}{k}}} ds.$$

The result is

$$\int_0^1 \int_0^1 \int_0^1 \left(\frac{x}{yz}\right)^{\frac{2\pi\phi e}{4\pi\phi e-1}} \left\{ 2\left(\frac{zy}{x}\right)^{\pi e\phi} \right\} dxdydz = 0.106598$$

We omit the calculations due to the obvious reasons.

3. Conclusion

1. Generalized fractional integral of the order n is obtained. The simplification of the C_n makes the recursive relation much more useful.

2. Questions about whether other forms of the generalized fractional integral are obtainable arise.

3. We checked all the numerical results with `Wolfram Alpha` in order to be sure.

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