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Triangle inequalities in inner-product spaces

Research Article

Martin Lukarevski[1*] **and Dan Stefan Marinescu**[2†]

1 Department of Mathematics and Statistics, University "Goce Delcev" - Stip, 2000, Stip, North Macedonia

2 Colegiul National "Iancu De Hunedoara", Hunedoara, Romania

Abstract: Tereshin's and Panaitopol's are known inequalities involving the median, circumradius and sides of the triangle. In this short note we generalize the inequalities to inner-product spaces. As an application we derive inequality for the median and the radius of the circumscribed sphere of an n-dimensional simplex.

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1. Introduction

Richard Bellman writes in [1] 'There are three reasons for the study of inequalities: practical, theoretical, and aesthetic'. The theory of geometric inequalities contains many beautiful inequalities and so justifies the third, aesthetic reason to study them. Such examples of triangle inequalities are Euler's inequality $R \geq 2r$ for the circumradius and the inradius, Weitzenböck's inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}K$, for the sum of the squares of the sides and the area K, Tsintsifas ineqality [7] $\frac{m_a}{w_a} \geq \frac{(b+c)^2}{4bc}$ for the ratio of the median and the angle bisector etc. For the median m_a we have the following chain of inequalities

$$\frac{b^2 + c^2}{4R} \leq m_a \leq \frac{Rs}{a} = \frac{bc}{4r} \leq \frac{(b+c)^2}{16r}.$$

The first is Tereshin's inequality, the second is Panaitopol's inequality. By $ah_a = 2K$, $K = rs$, Panaitopol's inequality can be rewritten as

$$\frac{R}{2r} \geq \frac{m_a}{h_a}.$$

* **Corresponding author**

* *E-mail:* *martin.lukarevski@ugd.edu.mk*
† *E-mail:* *marinescuds@gmail.com*

The aim of this paper is to extend Panaitopol's and Tereshin's inequality to inner-product spaces and to generalize these inequalities for a triangle.

2. Some preliminary remarks

Let the real or complex normed space $(X, ||\cdot||)$ be an inner-product space, that is, the norm comes from an inner-product. We present some result that we use in the next section.

Theorem 2.1.
(see [4]) Let $x_1, \dots, x_n \in X$, $n \geq 2$. For any $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $\alpha_1 + \cdots + \alpha_n = 1$, we have

$$\left\| x - \sum_{i=1}^{n} \alpha_i x_i \right\|^2 = \sum_{i=1}^{n} \alpha_i \|x - x_i\|^2 - \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2 \tag{1}$$

Theorem 2.2.
(Power of a point in inner-product spaces, see [5]) *Let $x_0, x_1, x_2 \in X$ such that $\|x_1 - x_0\| = \|x_2 - x_0\| = r$, $r \geq 0$. For $\alpha \in [0,1]$ let $w = \alpha x_1 + (1-\alpha)x_2$. Then*

$$\|w - x_1\| \cdot \|w - x_2\| = r^2 - \|w - x_0\|^2. \tag{2}$$

Theorem 2.3.
(see [5]) Let $y_0 \in X$, $r \geq 0$ and $x, x_1 \in X$ such that $\|x - y_0\| < r$, $\|x_1 - y_0\| = r$. Then there is unique pair (y_1, α) with $y_1 \in X$, $\alpha \in (0,1)$ such that

$$x = \alpha y_1 + (1-\alpha)x_1, \quad \|y_1 - y_0\| = r. \tag{3}$$

The following theorem is Ptolemy's inequality in inner-product spaces.

Theorem 2.4.
For all $x, y, z, t \in X$, it holds

$$\|x - y\| \cdot \|z - t\| + \|x - t\| \cdot \|y - z\| \geq \|x - z\| \cdot \|y - t\|. \tag{4}$$

Proof. See [2], [6]. □

3. The Panaitopol and Tereshin inequalities in inner-product spaces

Next we generalize the Panaitopol and Tereshin inequalities to inner-product spaces. The first result is generalization of Tereshin's inequality.

Theorem 3.1.
Let $y_0, x_0, x_1, \dots, x_n \in X$, $n \geq 2$ be distinct and such that $x_0, x_1, \dots, x_n \in S(y_0, r) = \{x \in X : \|x - y_0\| = r\}$. For $\alpha_1, \dots, \alpha_n \geq 0$ with $\alpha_1 + \cdots + \alpha_n = 1$ let $\overline{x} = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Then we have

$$2r\|x_0 - \overline{x}\| \geq \alpha_1 \|x_0 - x_1\|^2 + \cdots + \alpha_n \|x_0 - x_n\|^2. \tag{5}$$

Proof. By identity (1), we have

$$\begin{aligned}\|\overline{x}-y_0\|^2 &= \sum_{i=1}^{n}\alpha_i\|x_i-y_0\|^2-\sum_{1\le i<j\le n}\alpha_i\alpha_j\|x_i-x_j\|^2 \\ &< \sum_{i=1}^{n}\alpha_i r^2=r^2,\end{aligned} \tag{6}$$

so $\|\overline{x}-y_0\|<r$.
From Theorem 2.3 follows that for x_0 and $\overline{x}$ there is a pair (y_1,α), $y_1\in X$, $\alpha\in(0,1)$ such that

$$\overline{x}=\alpha y_1+(1-\alpha)x_0,\quad \|y_1-y_0\|=r.$$

We observe that as a consequence of the first equation we have

$$\|y_1-x_0\|=\|\overline{x}-y_1\|+\|\overline{x}-x_0\|.$$

Now we have

$$\begin{aligned}2r\|x_0-\overline{x}\| &\ge \|y_1-x_0\|\cdot\|x_0-\overline{x}\|=(\|\overline{x}-y_1\|+\|\overline{x}-x_0\|)\cdot\|x_0-\overline{x}\| \\ &= \|x_0-\overline{x}\|^2+\|\overline{x}-y_1\|\cdot\|\overline{x}-x_0\|\end{aligned} \tag{7}$$

By identity (1), we have

$$\|\overline{x}-x_0\|^2 = \sum_{i=1}^{n}\alpha_i\|x_i-x_0\|^2-\sum_{1\le i<j\le n}\alpha_i\alpha_j\|x_i-x_j\|^2 \tag{8}$$

From Theorem 2.2 and (6) follows with the assumption $\alpha_1+\cdots+\alpha_n=1$

$$\begin{aligned}\|\overline{x}-y_1\|\cdot\|\overline{x}-x_0\| &= r^2-\|\overline{x}-y_0\|^2 \\ &= r^2-\sum_{i=1}^{n}\alpha_i\|x_i-y_0\|^2+\sum_{1\le i<j\le n}\alpha_i\alpha_j\|x_i-x_j\|^2 \\ &= \sum_{1\le i<j\le n}\alpha_i\alpha_j\|x_i-x_j\|^2.\end{aligned} \tag{9}$$

Finally, by (7), (8) and (9) follows the desired inequality (5). □

Remark 3.1.

If $\alpha_1=\cdots=\alpha_n=\frac{1}{n}$, we obtain

$$2nr\cdot\left\|x_0-\frac{1}{n}(x_1+\cdots+x_n)\right\|\ge\|x_0-x_1\|^2+\cdots+\|x_0-x_n\|^2.$$

For $n=2$ that is Tereshin's inequality for triangle.

Remark 3.2.

Let A_i, $i=0,1,\dots n$ denote the vertices of an n-dimensional simplex and let R be the radius of the circumscribed sphere. Let G_0 be the centroid of the face opposite vertex A_0. Then we have

$$2nR\cdot A_0G_0\ge A_0A_1^2+\cdots+A_0A_n^2.$$

This inequality appears to be new for simplices. For tetrahedron $A_0A_1A_2A_3$ the inequality is

$$6R\cdot A_0G_0\ge A_0A_1^2+A_0A_2^2+A_0A_3^2,$$

see [3].

Next result is a generalization of Panaitopol's inequality to inner-product spaces.

Theorem 3.2.
Let $x, x_1, x_2, x_3 \in X$. Then we have

$$\begin{aligned} &\|2x_1 - x_2 - x_3\| \cdot \|x_2 - x_3\| \\ &\leq \|x - x_1\| \cdot \|x_2 - x_3\| + \|x - x_2\| \cdot \|x_1 - x_2\| + \|x - x_3\| \cdot \|x_1 - x_3\|. \end{aligned} \tag{10}$$

Proof. We consider the four elements in X: $x, x_2, x_2 + x_3 - x_1, x_3$ and apply Ptolemy's inequality (4) to obtain

$$\begin{aligned} &\|x - x_2\| \cdot \|x_2 - x_1\| + \|x - x_3\| \cdot \|x_3 - x_1\| \\ &\geq \|x + x_1 - x_2 - x_3\| \cdot \|x_2 - x_3\| \end{aligned} \tag{11}$$

On the other hand by triangle inequality we have

$$\begin{aligned} &\|x + x_1 - x_2 - x_3\| \cdot \|x_2 - x_3\| + \|x - x_1\| \cdot \|x_2 - x_3\| \\ &\geq \|2x_1 - x_2 - x_3\| \cdot \|x_2 - x_3\| \end{aligned} \tag{12}$$

Adding (11) and (12), we obtain the inequality (10). □

Remark 3.3.
If $x_1, x_2, x_3 \in S(x, R)$, then

$$\|2x_1 - x_2 - x_3\| \cdot \|x_2 - x_3\| \leq R\left(\|x_1 - x_2\| + \|x_2 - x_3\| + \|x_3 - x_1\|\right).$$

This is Panaitopol's inequality $am_a \leq Rs$ in inner-product spaces.

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