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Integral inequalities for generalized approximately h-convex functions on fractal sets via generalized local fractional integrals

Research Article

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Abstract: The present paper study the notion of generalized approximately h-convex function and define a novel generalized fractional integral operator on fractal sets. We derive local fractional integral inequalities involving newly defined generalized local fractional integral operator for generalized approximately h-convex function. Special cases of the results are also established.

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1. Introduction

The study of convex functions offered provoking results in multiple areas of mathematics e.g, mathematical economics, linear programming, optimization, dynamic systems and control theory. Nowadays, different kinds of integral inequalities concerning convex functions and their generalizations are getting attention of researchers to work on. More information can be found in the references [3–7, 15, 16, 19].

Definition 1.1.

A mapping $F : I \subseteq R \to R$ is known to be convex, if

$$F(\ell r + (1-\ell)s) \leq \ell F(r) + (1-\ell)F(s) \tag{1}$$

If $\forall\ r, s \in I,\ \ell \in [0,1]$

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Famous inequality derived by J.Hadamard [9] in 1881 is known as Hermite-Hadamard Inequality. Let $F : I \subseteq R \to R$, be a convex function if $r, s \in I$ and $r < s$, then

$$F\Big(\frac{r+s}{2}\Big) \leq \frac{1}{s-r}\int_r^s F(y)dy \leq \frac{F(r)+F(s)}{2} \tag{2}$$

S. Varosanec [26] presented the notion of $h-$convex functions.

Definition 1.2.

Let $h : J \subseteq R$ be non negative function and $h \neq 0$. A non negative function $F : I = [r, s] \to R$ is known to be h-convex, if for all $\ell \in [0, 1]$.

$$F(\ell r + (1-\ell)s) \leq h(\ell)F(r) + h(1-\ell)F(s) \tag{3}$$

In [12] Kashuri et al defined approximately h-convex function. Consider $(X, ||.||_I)$ be a normed quasi linear space, and I is nonempty convex subset of $X, d : X \times X \to R$.

Definition 1.3.

Consider $h : [0, 1] \to R$ be non negative function and $h \neq 0$. A function $F : I \to R$ is called to be approximately h-convex, if

$$F(\ell r + (1-\ell)s) \leq h(\ell)F(r) + h(1-\ell)F(s) + d(r, s) \tag{4}$$

holds for all $\ell \in (0, 1)$ and $r, s \in I$

Sarikaya and Ertuğral in [17] presented the left-sided and right-sided generalized fractional integral operators, as:

$$_{r^+}I_{\varphi}F() = \int_r \frac{\varphi(-\ell)}{(-\ell)}F(\ell)d\ell, \ \ > r \tag{5}$$

$$_{s^-}I_{\varphi}F() = \int^s \frac{\varphi(\ell-)}{(\ell-)}F(\ell)d\ell, \ \ < s \tag{6}$$

Hadamard inequality for the generalized fractional integral operators is established by Sarikaya and Ertuğral in [17].

Theorem 1.1.

Let $F : [r, s] \to R$ be a convex function on $[r, s]$, with $r < s$, then for fractional integral operators the following inequality holds:

$$F\left(\frac{r+s}{2}\right) \leq \frac{1}{2\Lambda(1)}[_{r^+}I_{\varphi}F(s) + {_{s^-}I_{\varphi}}F(r)] \leq \frac{F(r)+F(s)}{2} \tag{7}$$

where the mapping $\Lambda(x) : [0, 1] \to R$ is given by

$$\Lambda() := \int_0 \frac{\varphi((s-r)\ell)}{\ell}d\ell \tag{8}$$

Kashuri et al. [11], established inequalities of Hermite Hadamard type for approximately h-convex functions involving generalized fractional integrals.

Theorem 1.2.
Let $F : [r, s] \to R$ be an approximately h-convex on $[r, s]$, with $r < s$. Then for generalized fractional integrals the following inequality holds:

$$\frac{1}{2h(\frac{1}{2})}F\left(\frac{r+s}{2}\right) - A_1 \leq \frac{1}{2\Lambda(1)}[_{r^+}I_\varphi F(s) + {}_{s^-}I_\varphi F(r)]$$
$$\leq \frac{[F(r)+F(s)]}{2\Lambda(1)}\int_0^1 \frac{\varphi((s-r)\ell)}{\ell}[h(\ell)+h(1-\ell)]d\ell + d(r,s)$$

where,

$$A_1 := \frac{1}{2\Lambda(1)h(\frac{1}{2})}\int_r^s \frac{\varphi(s-)}{s-}d(,r+s-)d.$$

We now discuss some preliminaries of local fractional calculus theory introduced by Yang in [28, 29]. These concepts and consequences are linked with fractal order derivatives and integrals.

If $r_1^\varsigma, r_2^\varsigma, r_3^\varsigma \in R^\varsigma$ $(0 < \varsigma \leq 1)$, then

- $r_1^\varsigma + r_2^\varsigma \in R^\varsigma, r_1^\varsigma r_2^\varsigma \in R^\varsigma$,
- $r_1^\varsigma + r_2^\varsigma = r_2^\varsigma + r_1^\varsigma = (r_1 + r_2)^\varsigma = (r_2 + r_1)^\varsigma$,
- $r_1^\varsigma + (r_2^\varsigma + r_3^\varsigma) = (r_1 + r_2)^\varsigma + r_3^\varsigma$,
- $r_1^\varsigma r_2^\varsigma = r_2^\varsigma r_1^\varsigma = (r_1 r_2)^\varsigma = (r_2 r_1)^\varsigma$,
- $r_1^\varsigma(r_2^\varsigma r_3^\varsigma) = (r_1^\varsigma r_2^\varsigma)r_3^\varsigma$,
- $r_1^\varsigma(r_2^\varsigma + r_3^\varsigma) = r_1^\varsigma r_2^\varsigma + r_1^\varsigma r_3^\varsigma$,
- $r_1^\varsigma + 0^\varsigma = 0^\varsigma + r_1^\varsigma = r_1^\varsigma$, and $r_1^\varsigma 1^\varsigma = 1^\varsigma r_1^\varsigma = r_1^\varsigma$
- If $r_1^\varsigma < r_2^\varsigma$, then $r_1^\varsigma + r_3^\varsigma < r_2^\varsigma + r_3^\varsigma$,
- If $0^\varsigma < r_1^\varsigma, 0^\varsigma < r_2^\varsigma$, then $0^\varsigma < r_1^\varsigma . r_2^\varsigma$,

Local fractional derivative and integral on R^ς are defined as,

Definition 1.4.
([28, 29]) A non-differentiable function $F : R \to R^\varsigma, y \to F(y)$ is local fractional continuous at y_0, if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|F(y) - F(y_0)| < \epsilon^\varsigma$$

holds $|y - y_0| < \delta$, with $\epsilon, \delta \in R$. If $F(y)$ is local continuous on (c, d), and we donate $F(y) \in C_\varsigma(c, d)$.

Definition 1.5.
([28, 29]) Local fractional derivative of the function $F(y)$ of order ς at $y = y_0$ can be defined as

$$F^{(\varsigma)}(y_0) = \left.\frac{d^\varsigma F(y)}{d\ell^\varsigma}\right|_{y=y_0} = \lim_{y \to y_0} \frac{\Gamma(1+\varsigma)(F(y) - F(y_0))}{(y - y_0)}$$

$D_\varsigma(b,c)$ is ς-local derivative set. If there exists $F^{((K+1)\varsigma)}(y) = \overbrace{D_y^\varsigma ... D_y^\varsigma}^{(n+1)times} F(y)$ for any $y \in I \subseteq R$, we denote $F \in D_{(n+1)\varsigma}(I)$, and $n = 0, 1, 2, ...$

Definition 1.6.
([28, 29]) Let $F(y) \in C_\varsigma[c,d]$. Local fractional integral of $F(w)$ can be defined by

$$ {}_bI_c^\varsigma F(y) = \frac{1}{\Gamma(1+\varsigma)} \int_b^c F(\ell)(d\ell)^\varsigma = \frac{1}{\Gamma(1+\varsigma)} \lim_{\Delta\ell \to 0} \sum_{f=0}^{N-1} F(\ell_f)(\Delta\ell_f)^\varsigma, $$

Where $c = \ell_0 < \ell_1 < ... < \ell_{N-1} < \ell_N = d, [\ell_f, \ell_{f+1}]$ is partition of $[c,d]$, $\Delta\ell_f = \Delta\ell_{f+1} - \Delta\ell_f, \Delta\ell = max\{\ell_0, \ell_1 ... \ell_{N-1}\}$.

Note that ${}_cI_c^\varsigma F(y) = 0$ and ${}_cI_d^\varsigma F(y) = -{}_dI_c^\varsigma F(y)$ if $c < d$. We denote $F(y) \in I_y^\varsigma[c,d]$ if there exists${}_bI_y^\varsigma F(y)$ for any $y \in [b,c]$.

Lemma 1.1.
([28, 29])

1. *Let $g(y) = f^{(\varsigma)}(y) \in C_\varsigma[c,d]$, then*
$$ {}_bI_c^\varsigma g(y) = f(d) - f(c) $$
2. *Let $g(y), f(y) \in D_\varsigma[c,d]$ and $g^{(\varsigma)}(y), f^{(\varsigma)}(y) \in C_\varsigma[b,c]$, then*
$$ {}_bI_c^\varsigma g(y) f^{(\varsigma)}(y) = g(y)f(y)|_b^c - {}_bI_c^\varsigma g^{(\varsigma)}(y) f(y). $$

Lemma 1.2.
([28, 29])

$$ \frac{d^\varsigma y^{s\varsigma}}{du^\varsigma} = \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s-1)\varsigma)} y^{(s-1)\varsigma}; $$
$$ \frac{1}{\Gamma(\varsigma+1)} \int_b^c y^{s\varsigma}(du)^\varsigma = \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} (d^{(s+1)\varsigma} - c^{(s+1)\varsigma}), \quad s > 0 $$

Lemma 1.3.
([28, 29])

$$ {}_bI_c^\varsigma 1^\varsigma = \frac{(d-c)^\varsigma}{\Gamma(1+\varsigma)} $$

Lemma 1.4.
(([28, 29]) (Generalized Hölder's inequality) Let $p, q > 1$ with $p^{-1} + q^{-1} = 1$, let $g(w), f(w) \in C_\varsigma[c,d]$,Then

$$ \frac{1}{\Gamma(\varsigma+1)} \int_b^c |g(y)f(y)|(dy)^\varsigma \le \left(\frac{1}{\Gamma(\varsigma+1)} \int_b^c |g(y)|^p (dy)^\varsigma \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\varsigma+1)} \int_b^c |f(y)|^q (dy)^\varsigma \right)^{\frac{1}{q}} $$

Recall generalized beta function:

$$B_\varsigma(y,x)=\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \ell^{(y-1)\varsigma}(1-\ell)^{(x-1)\varsigma}(d\ell)^\varsigma,\quad y>0,x>0 \tag{9}$$

Local fractional theory has solid applications in control theory, communication engineering, random walk process and Physics [1, 13, 27, 30]. Many researchers studied various types of integral inequalities for generalized definitions of convexity on fractal sets.[2, 8, 10, 14, 18]. Wenbing Sun established various types of Integral inequalities for different generalizations of convex functions in fractal theory [20–25]. According to our knowledge the study of approximately $h-$convex functions has not been carried out in fractal domain. In this research paper we study the concept of generalized approximately $h-$convex functions on fractal sets, we also present a novel fractional integral operator on fractal sets through which we establish generalized local fractional integral inequalities.

2. Main Results

We define a function $\varphi_\beta:[0,\infty)\to[0,\infty)$ satisfying the following condition, if for $X^\varsigma, Y^\varsigma, Z^\varsigma>0$ and independence of $a^\varsigma, b^\varsigma>0$

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \frac{\varphi_\varsigma(\ell^\varsigma)}{\ell^\varsigma}(d\ell)^\varsigma<+\infty\\
&\frac{1}{X^\varsigma}\le\frac{\varphi_\varsigma(b^\varsigma)}{\varphi_\varsigma(a^\varsigma)}\le X^\varsigma,\quad for\quad \frac{1}{2^\varsigma}\le\frac{b^\varsigma}{a^\varsigma}\le 2^\varsigma\\
&\frac{\varphi_\varsigma(a^\varsigma)}{a^{2\varsigma}}\le Y^\varsigma\frac{\varphi_\varsigma(b^\varsigma)}{b^{2\varsigma}}\quad for\quad b^\varsigma<a^\varsigma\\
&\left|\frac{\varphi_\varsigma(a^\varsigma)}{a^{2\varsigma}}-\frac{\varphi_\varsigma(b^\varsigma)}{b^{2\varsigma}}\right|\le Z^\varsigma|a^\varsigma-b^\varsigma|\frac{\varphi_\varsigma(a^\varsigma)}{a^{2\varsigma}}\quad for\quad \frac{1}{2^\varsigma}\le\frac{b^\varsigma}{a^\varsigma}\le 2^\varsigma
\end{aligned}$$

Now we define generalized local fractional integral operators,

$$_{r^+}I^\varsigma_\varphi F()=\frac{1}{\Gamma(1+\varsigma)}\int_r \frac{\varphi_\varsigma(-\ell)^\varsigma}{(-\ell)^\varsigma}F(\ell)(d\ell)^\varsigma,\ >r \tag{10}$$

$$_{s^-}I^\varsigma_\varphi F()=\frac{1}{\Gamma(1+\varsigma)}\int^s \frac{\varphi_\varsigma(\ell-)^\varsigma}{(\ell-)^\varsigma}F(\ell)(d\ell)^\varsigma,\ <s \tag{11}$$

Definition 2.1.
Suppose $h:J\to R$ be non negative function and $h^\varsigma\not\equiv 0$. A function $F:I\to R^\varsigma(0<\varsigma\le 1)$ be ς order fractal dimensional. If F be non-negative $(F\ge 0^\varsigma)$ and for $r,s\in I$ and $\ell\in(0,1)$, then

$$F(\ell r+(1-\ell)s)\le h^\varsigma(\ell)F(r)+h^\varsigma(1-\ell)F(s)+d^\varsigma(r,s) \tag{12}$$

is called to be generalized approximately h-convex function on fractal sets.

Remark 2.1.
If $h^\varsigma(\ell)=\ell^\varsigma$, the generalized approximately h-convex function turned into generalized approximately convex function. If $h^\varsigma(\ell)=\ell^\varsigma(1-\ell)^\varsigma$, $h^\varsigma(\ell)=1$, $h^\varsigma(\ell)=\ell^{s\varsigma}$, the generalized approximately h- convex function turned into generalized approximately tgs-convex function, generalized approximately P-convex function, generalized approximately s-convex function respectively.

Theorem 2.1.
Let $F : [r,s] \to R^\varsigma$ and $F() \in I^\varsigma[r,s]$ be an approximately h-convex on $[r,s]$, with $0 \leq r < s$. Then following generalized local fractional integrals inequalities hold:

$$\frac{1}{2^\varsigma h^\varsigma(\frac{1}{2})} F\left(\frac{r+s}{2}\right) - B_1 \leq \frac{1}{2^\varsigma \Lambda^\varsigma(1)} [{}_{r^+}I_\varphi^\varsigma F(s) + {}_{s^-}I_\varphi^\varsigma F(r)]$$
$$\leq \frac{[F(r)+F(s)]}{2^\varsigma \Lambda^\varsigma(1)} \frac{1}{\Gamma(1+\varsigma)} \int_0^1 \frac{\varphi_\varsigma((s-r)\ell)^\varsigma}{\ell^\varsigma} [h^\varsigma(\ell) + h^\varsigma(1-\ell)](d\ell)^\varsigma + d^\varsigma(r,s)$$

where,

$$B_1 := \frac{1}{2\Lambda^\varsigma(1) h^\varsigma(\frac{1}{2})\Gamma(1+\varsigma)} \int_r^s \frac{\varphi_\varsigma(s-)^\varsigma}{(s-)^\varsigma} d^\varsigma(,r+s-)(d)^\varsigma.$$

Proof. As F is generalized approximately h-convex function, so we have following inequality

$$F\left(\frac{u+v}{2}\right) \leq h^\varsigma(\frac{1}{2})[F(u)+F(v)] + d^\varsigma(u,v) \tag{13}$$

Taking $u = \ell r + (1-\ell)s$ and $v = \ell s + (1-\ell)r$

$$\frac{1}{h^\varsigma(\frac{1}{2})} F\left(\frac{r+s}{2}\right) \leq F(\ell r + (1-\ell)s)$$
$$+F(\ell s + (1-\ell)r) + \frac{1}{h^\varsigma(\frac{1}{2})} d(\ell r + (1-\ell)s, \ell s + (1-\ell)r)$$

Multiply both sides of above inequality by $\frac{\varphi((s-r)\ell)}{\ell}$ and integrate local fractionally with respect to ℓ over $[0,1]$, we have

$$\frac{1}{h^\varsigma(\frac{1}{2})} F\left(\frac{r+s}{2}\right) \frac{1}{\Gamma(1+\varsigma)} \int_0^1 \frac{\varphi((s-r)\ell)}{\ell} (d\ell)^\varsigma$$
$$\leq \frac{1}{\Gamma(1+\varsigma)} \int_0^1 \frac{\varphi((s-r)\ell)}{\ell} F(\ell r + (1-\ell)s)(d\ell)^\varsigma +$$
$$\frac{1}{\Gamma(1+\varsigma)} \int_0^1 \frac{\varphi((s-r)\ell)}{\ell} F(\ell s + (1-\ell)v)(d\ell)^\varsigma$$
$$+\frac{1}{h^\varsigma(\frac{1}{2})\Gamma(1+\varsigma)} \int_0^1 \frac{\varphi((s-r)\ell)}{\ell} d(\ell r + (1-\ell)s, \ell s + (1-\ell)r)(d\ell)^\varsigma$$

As a consequence, we get

$$\frac{1}{2^\varsigma h^\varsigma(\frac{1}{2})} F\left(\frac{r+s}{2}\right) \leq \frac{1}{2^\varsigma \Lambda^\varsigma(1)} [{}_{r^+}I_\varphi^\varsigma F(s) + {}_{s^-}I_\varphi^\varsigma F(r)]$$
$$+\frac{1}{2^\varsigma \Lambda^\varsigma(1) h^\varsigma(\frac{1}{2})\Gamma(1+\varsigma)} \int_r^s \frac{\varphi_\varsigma(s-)^\varsigma}{(s-)^\varsigma} d^\varsigma(,r+s-)(d)^\varsigma.$$

we get the proof of right hand side of the inequality.

To prove left hand side of the inequality, since F is approximately h-convex, we have

$$F(\ell r + (1-\ell)s) + F(\ell s + (1-\ell)v) \leq [F(r)+F(s)][h^\varsigma(\ell) + h^\varsigma(1-\ell)] + 2^\varsigma d^\varsigma(r,s)$$

Multiply both sides of inequality by $\frac{\varphi((s-r)\ell)}{\ell}$ and integrating local fractionally with respect to ℓ over $[0,1]$, we have,

$$\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \frac{\varphi((s-r)\ell)}{\ell}F(\ell u+(1-\ell)v)(d\ell)^\varsigma + \frac{1}{\Gamma(1+\varsigma)}\int_0^1 \frac{\varphi((s-r)\ell)}{\ell}F(\ell v+(1-\ell)u)(d\ell)^\varsigma + 2^\varsigma d^\varsigma(r,s)\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \frac{\varphi((s-r)\ell)}{\ell}(d\ell)^\varsigma$$
$$\leq [F(r)+F(s)]\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \frac{\varphi((s-r)\ell)}{\ell}[h^\varsigma(\ell)+h^\varsigma(1-\ell)](d\ell)^\varsigma + 2^\varsigma d^\varsigma(r,s)\Lambda^\varsigma(1)$$

As a result, we get

$$\frac{1}{2^\varsigma\Lambda^\varsigma(1)}[{}_{r^+}I_\varphi^\varsigma F(s) + {}_{s^-}I_\varphi^\varsigma F(r)]$$
$$\leq \frac{[F(r)+F(s)]}{2^\varsigma\Lambda^\varsigma(1)\Gamma(1+\varsigma)}\int_0^1 \frac{\varphi((s-r)\ell)}{\ell}[h^\varsigma(\ell)+h^\varsigma(1-\ell)](d\ell)^\varsigma + d^\varsigma(r,s)$$

□

Corollary 2.1.
If $\varphi(\ell)=\ell^\varsigma$ then following inequality holds

$$\frac{1}{2^\varsigma h^\varsigma(\frac{1}{2})}F\left(\frac{r+s}{2}\right) - B_1 \leq \frac{1}{2^\varsigma(s-r)^\varsigma}[{}_{r^+}I^\varsigma F(s) + {}_{s^-}I^\varsigma F(r)]$$
$$\leq \frac{[F(r)+F(s)]}{2^\varsigma}\frac{1}{\Gamma(1+\varsigma)}\int_0^1[h^\varsigma(\ell)+h^\varsigma(1-\ell)](d\ell)^\varsigma + \frac{d^\varsigma(r,s)}{(s-r)^\varsigma}$$

where,

$$B_1 := \frac{1}{2^\varsigma h^\varsigma(\frac{1}{2})\Gamma(1+\varsigma)(s-r)^\varsigma}\int_r^s d^\varsigma(,r+s-)(d)^\varsigma.$$

Corollary 2.2.
If $h(\ell)=\ell^\varsigma$ in above inequality then following inequality holds

$$\frac{1}{2^\varsigma h^\varsigma(\frac{1}{2})}F\left(\frac{r+s}{2}\right) - B_1 \leq \frac{1}{2^\varsigma(s-r)^\varsigma}[{}_{r^+}I^\varsigma F(s) + {}_{s^-}I^\varsigma F(r)]$$
$$\leq [F(r)+F(s)]\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)} + \frac{d^\varsigma(r,s)}{(s-r)^\varsigma}$$

Corollary 2.3.
If $h(\ell)=\ell^\varsigma(1-\ell)^\varsigma$ in above inequality then following inequality holds

$$\frac{1}{2^\varsigma h^\varsigma(\frac{1}{2})}F\left(\frac{r+s}{2}\right) - B_1 \leq \frac{1}{2^\varsigma(s-r)^\varsigma}[{}_{r^+}I^\varsigma F(s) + {}_{s^-}I^\varsigma F(r)]$$
$$\leq [F(r)+F(s)]\beta_\varsigma(2,2) + \frac{d^\varsigma(r,s)}{(s-r)^\varsigma}$$

Corollary 2.4.
If $h(\ell)=1^\varsigma$ in above inequality then following inequality holds

$$\frac{1}{2^\varsigma h^\varsigma(\frac{1}{2})}F\left(\frac{r+s}{2}\right) - B_1 \leq \frac{1}{2^\varsigma(s-r)^\varsigma}[{}_{r^+}I^\varsigma F(s) + {}_{s^-}I^\varsigma F(r)]$$
$$\leq \frac{[F(r)+F(s)]}{\Gamma(1+\varsigma)} + \frac{d^\varsigma(r,s)}{(s-r)^\varsigma}$$

Corollary 2.5.
If $h(\ell)=\ell^{s\varsigma}$ in above inequality then following inequality holds

$$\frac{1}{2^{\varsigma}h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)-B_1\leq\frac{1}{2^{\varsigma}(s-r)^{\varsigma}}[{}_{r^+}I^{\varsigma}F(s)+{}_{s^-}I^{\varsigma}F(r)]$$
$$\leq[F(r)+F(s)]\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(1+s)\varsigma)}+\frac{d^{\varsigma}(r,s)}{(s-r)^{\varsigma}}$$

Theorem 2.2.
Let $F:[r,s]\to R^{\varsigma}$ and $F()\in I^{\varsigma}[r,s]$ be an approximately h-convex on $[r,s]$, with $0\leq r<s$. Then following generalized local fractional integrals inequalities hold:

$$\frac{1}{2^{\varsigma}h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)-B_2\leq\frac{1}{2^{\varsigma}\Lambda^{\varsigma}(1)}[{}_{r^+}I^{\varsigma}_{\varphi}F(s)+{}_{s^-}I^{\varsigma}_{\varphi}F(r)]$$
$$\leq\frac{[F(r)+F(s)]}{2^{\varsigma}\Lambda^{\varsigma}(1)}\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi_{\varsigma}((s-r)\ell)^{\varsigma}}{\ell^{\varsigma}}[h^{\varsigma}(\frac{\ell}{2})+h^{\varsigma}(\frac{2-\ell}{2})](d\ell)^{\varsigma}+d^{\varsigma}(r,s)$$

where,

$$B_2:=\frac{1}{2^{\varsigma}\Lambda^{\varsigma}(1)h^{\varsigma}(\frac{1}{2})\Gamma(1+\varsigma)}\int_{\frac{r+s}{2}}^{s}\frac{\varphi_{\varsigma}(s-)^{\varsigma}}{(s-)^{\varsigma}}d^{\varsigma}(,r+s-)(d)^{\varsigma}.$$

Proof. As F is generalized approximately h-convex function, so we have following inequality

$$F\left(\frac{u+v}{2}\right)\leq h^{\varsigma}(\frac{1}{2})[F(u)+F(v)]+d^{\varsigma}(u,v)\tag{14}$$

Taking $u=\frac{\ell}{2}r+\frac{(2-\ell)}{2}s$ and $v=\frac{\ell}{2}s+\frac{(2-\ell)}{2}r$

$$\frac{1}{h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)\leq F\left(\frac{\ell}{2}r+\frac{(2-\ell)}{2}s\right)$$
$$+F\left(\frac{\ell}{2}s+\frac{(2-\ell)}{2}r\right)+\frac{1}{h^{\varsigma}(\frac{1}{2})}d\left(\frac{\ell}{2}r+\frac{(2-\ell)}{2}s,\frac{\ell}{2}s+\frac{(2-\ell)}{2}r\right)$$

Multiply both sides of the inequality by $\frac{\varphi((s-r)\ell)}{\ell}$ and integrating local fractionally with respect to ℓ over $[0,1]$, we obtain

$$\frac{1}{h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}(d\ell)^{\varsigma}$$
$$\leq\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}F\left(\frac{\ell}{2}r+\frac{(2-\ell)}{2}s\right)(d\ell)^{\varsigma}+\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}F\left(\frac{\ell}{2}s+\frac{(2-\ell)}{2}r\right)(d\ell)^{\varsigma}$$
$$+\frac{1}{h^{\varsigma}(\frac{1}{2})\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}d\left(\frac{\ell}{2}r+\frac{(2-\ell)}{2}s,\frac{\ell}{2}s+\frac{(2-\ell)}{2}r)\right)(d\ell)^{\varsigma}$$

As a consequence, we get

$$\frac{1}{2^{\varsigma}h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)\leq\frac{1}{2^{\varsigma}\Lambda^{\varsigma}(1)}[{}_{(\frac{r+s}{2})^+}I^{\varsigma}_{\varphi}F(s)+{}_{(\frac{r+s}{2})^-}I^{\varsigma}_{\varphi}F(r)]$$
$$+\frac{1}{2^{\varsigma}\Lambda^{\varsigma}(1)h^{\varsigma}(\frac{1}{2})\Gamma(1+\varsigma)}\int_{(\frac{r+s}{2})}^{s}\frac{\varphi_{\varsigma}(s-)^{\varsigma}}{(s-)^{\varsigma}}d^{\varsigma}(,r+s-)(d)^{\varsigma}.$$

we proved the right hand side of the inequality.

To prove left hand side of the inequality, since F is approximately h-convex, we have

$$F\left(\frac{\ell}{2}r+\frac{(2-\ell)}{2}s\right)+F\left(\frac{\ell}{2}s+\frac{(2-\ell)}{2}r\right)\leq[F(r)+F(s)]\left[h^{\varsigma}\left(\frac{\ell}{2}\right)+h^{\varsigma}\left(\frac{2-\ell}{2}\right)\right]+2^{\varsigma}d^{\varsigma}(r,s)$$

Multiplying both sides of inequality by $\frac{\varphi((s-r)\ell)}{\ell}$ and integrating local fractionally with respect to ℓ over $[0,1]$, we obtain

$$\begin{aligned}&\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}F\left(\frac{\ell}{2}r+\frac{(2-\ell)}{2}s\right)(d\ell)^{\varsigma}\\&+\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}F\left(\frac{\ell}{2}s+\frac{(2-\ell)}{2}r\right)(d\ell)^{\varsigma}+2^{\varsigma}d^{\varsigma}(r,s)\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}(d\ell)^{\varsigma}\\&\leq[F(r)+F(s)]\frac{1}{\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}\left[h^{\varsigma}\left(\frac{\ell}{2}\right)+h^{\varsigma}\left(\frac{2-\ell}{2}\right)\right](d\ell)^{\varsigma}+2^{\varsigma}d^{\varsigma}(r,s)\Lambda^{\varsigma}(1)\end{aligned}$$

As a result, we get

$$\begin{aligned}&\frac{1}{2^{\varsigma}\Lambda^{\varsigma}(1)}\left[{}_{(\frac{r+s}{2})^+}I_{\varphi}^{\varsigma}F(s)+{}_{(\frac{r+s}{2})^-}I_{\varphi}^{\varsigma}F(r)\right]\\&\leq\frac{[F(r)+F(s)]}{2^{\varsigma}\Lambda^{\varsigma}(1)\Gamma(1+\varsigma)}\int_0^1\frac{\varphi((s-r)\ell)}{\ell}\left[h^{\varsigma}\left(\frac{\ell}{2}\right)+h^{\varsigma}\left(\frac{2-\ell}{2}\right)\right](d\ell)^{\varsigma}+d^{\varsigma}(r,s)\end{aligned}$$

□

Corollary 2.6.
If $\varphi(\ell)=\ell^{\varsigma}$ then following inequality holds

$$\begin{aligned}&\frac{1}{2^{\varsigma}h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)-B_2\leq\frac{1}{2^{\varsigma}(s-r)^{\varsigma}}\left[{}_{(\frac{r+s}{2})^+}I^{\varsigma}F(s)+{}_{(\frac{r+s}{2})^-}I^{\varsigma}F(r)\right]\\&\leq\frac{[F(r)+F(s)]}{2^{\varsigma}}\frac{1}{\Gamma(1+\varsigma)}\int_0^1\left[h^{\varsigma}\left(\frac{\ell}{2}\right)+h^{\varsigma}\left(\frac{2-\ell}{2}\right)\right](dt)^{\varsigma}+\frac{d^{\varsigma}(r,s)}{(s-r)^{\varsigma}}\end{aligned}$$

where,

$$B_2:=\frac{1}{2^{\varsigma}h^{\varsigma}(\frac{1}{2})\Gamma(1+\varsigma)(s-r)^{\varsigma}}\int_{(\frac{r+s}{2})}^{s}d^{\varsigma}(,r+s-)(d)^{\varsigma}.$$

Corollary 2.7.
If $h(\ell)=\ell^{\varsigma}$ in above inequality then following inequality holds

$$\begin{aligned}&\frac{1}{2^{\varsigma}h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)-B_2\leq\frac{1}{2^{\varsigma}(s-r)^{\varsigma}}\left[{}_{(\frac{r+s}{2})^+}I^{\varsigma}F(s)+{}_{(\frac{r+s}{2})^-}I^{\varsigma}F(r)\right]\\&\leq\frac{[F(r)+F(s)]}{\Gamma(1+\varsigma)}+\frac{d^{\varsigma}(r,s)}{(s-r)^{\varsigma}}\end{aligned}$$

Corollary 2.8.
If $h(\ell)=\ell^{\varsigma}(1-\ell)^{\varsigma}$ in above inequality then following inequality holds

$$\begin{aligned}&\frac{1}{2^{\varsigma}h^{\varsigma}(\frac{1}{2})}F\left(\frac{r+s}{2}\right)-B_2\leq\frac{1}{2^{\varsigma}(s-r)^{\varsigma}}\left[{}_{(\frac{r+s}{2})^+}I^{\varsigma}F(s)+{}_{(\frac{r+s}{2})^-}I^{\varsigma}F(r)\right]\\&\leq[F(r)+F(s)]\left[\frac{\beta_{\varsigma}(2,2)}{2^{\varsigma}}+\frac{\Gamma(1+\varsigma)}{4^{\varsigma}\Gamma(1+\varsigma)}-\frac{\Gamma(1+2\varsigma)}{8^{\varsigma}\Gamma(1+3\varsigma)}\right]+\frac{d^{\varsigma}(r,s)}{(s-r)^{\varsigma}}\end{aligned}$$

Corollary 2.9.
If $h(\ell) = 1^{\varsigma}$ in above inequality then following inequality holds

$$\frac{1}{2^{\varsigma} h^{\varsigma}(\frac{1}{2})} F\left(\frac{r+s}{2}\right) - B_2 \leq \frac{1}{2^{\varsigma}(s-r)^{\varsigma}} [{}_{(\frac{r+s}{2})^+} I^{\varsigma} F(s) + {}_{(\frac{r+s}{2})^-} I^{\varsigma} F(r)]$$
$$\leq \frac{[F(r)+F(s)]}{\Gamma(1+\varsigma)} + \frac{d^{\varsigma}(r,s)}{(s-r)^{\varsigma}}$$

Corollary 2.10.
If $h(\ell) = \ell^{s\varsigma}$ in above inequality then following inequality holds

$$\frac{1}{2^{\varsigma} h^{\varsigma}(\frac{1}{2})} F\left(\frac{r+s}{2}\right) - B_2 \leq \frac{1}{2^{\varsigma}(s-r)^{\varsigma}} [{}_{(\frac{r+s}{2})^+} I^{\varsigma} F(s) + {}_{(\frac{r+s}{2})^-} I^{\varsigma} F(r)]$$
$$\leq \frac{[F(r)+F(s)]}{\Gamma(1+\varsigma)} + \frac{d^{\varsigma}(r,s)}{(s-r)^{\varsigma}}$$

3. Conclusion

This study presented the concept of generalized approximately h-convex function on fractal sets. Involving generalized local fractional integral operators, we proved some latest generalized local fractional integral inequalities. Some special cases of novel findings are also established. It is to be expected that the the notion introduced in this study may open new doors for researchers to work on generalized approximately h-convex function in different directions.

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