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On fractional variable-order neural networks with time-varying external inputs

Research Article

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Abstract: This research discuss the existence, uniqueness, asymptotic stability, and global asymptotic synchronization of a class of Caputo variable-order neural networks with time-varying external inputs. Theory of contraction mapping is used to establish a sufficient condition for determining the existence and uniqueness of the equilibrium point. Using the variable fractional Lyapunov approach, we investigate the asymptotic stability of the unique equilibrium. Synchronization of variable-order chaotic networks is also studied using an effective controller. Three numerical examples are provided to show the efficacy of the results obtained.

Keywords: Fractional variable-order calculus • Fractional variable-order neural networks • Asymptotic stability • Global Asymptotic synchronization

1. Introduction

Fractional calculus is an old mathematical concept that was created long ago by mathematicians such as Leibniz, Liouville, Riemann, and others. However, it did not attract much attention intil the past few dacates. Due to the complexity of calculation and the ambiguity of its geometric importance researchers discovered that fractional calculus can precisely explain several anomalous events, and it is now widely utilized to describe various mathematical issues in science and engineering [3][16][21] [18] [22].

However, even though the constant fractional calculus concept may handle certain extremely significant physical issues, it cannot capture major classes of physical events in which the order is a function of either dependent or independent variables. As a result, it implies that there are categories of physical problems that are better characterized by variable-order operators [9][4]. [24], introduced the first variable fractional order operator concept.

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In contrast, various authors have provided definitions for variable fractional order differential operators, each with a distinct meaning to meet the objectives. [20][1]. In [26], the definitions of variable fractional order operators of the Riemann– Liouville, Caputo, and Coimbra types were compared.

Fractional-order neural networks have received a lot of attention recently [17][13][19], and they play key roles as tools not just in physics, but also in control systems and engineering. The incorporation of fractional calculus into neural networks is a fresh and fascinating idea. The main benefit of using variable fractional calculus in such systems is to describe more accurately the behavior of the system and because it has adaptive memory for previous encounters.

One of the core concerns in control theory is the asymptotical stability analysis of fractional order systems, which seeks to identify certain stability criteria under which systems are asymptotically stable. The Lyapunov method is a classic method to dealing with the stability problem in nonlinear fractional order systems. Due to the memory effect and the weakly singular kernels of the fractional order derivative, the fractional Lyapunov approach, which differs from the conventional Lyapunov method, was not created until 2009 in [27] and 2010 in [28], and its application was not accessible until 2014 [6][15][7]. Fractional variable-order neural network does not fall into this category, to the best of our knowledge, there are few clearly verifiable asymptotical stability criteria for fractional variable-order systems [14].

Synchronization has been an important study topic in nonlinear science and has been extensively studied in a variety of academic domains [12][2][10]. It has been established that some neural networks can display chaotic behavior, thus, there have been many synchronization results and methods about constant fractoinal order neural networks in the last decade [11][25][5][29][30][31]. These results and methods could not be easily extended and applied to the variable order case due to the complex dynamic of variable-order systems, and there are few theoretical findings on the synchronization of variable-order systems. As a result, developing some theoretically adequate conditions for synchronization of variable-order neural networks is both required and difficult.

In this work, we focus on the existence, uniqueness, asymptotic stability and global asymptotic synchronization analysis of fractional variable-order neural networks with time-varying external inputs, where the variable-order fractional derivatives is in Caputo meaning.

The rest of this work may be found below. Section 2 introduces several Definitions, and Theorems for fractional variable calculus. Section 3.1 includes a suitable condition for ensuring the existence and uniqueness of the equilibrium point. Section 3.2 proposes the asymptotic stability of the model's equilibruim point. The global asymptotic synchronization of the suggested network is addressed in Section 3.3. Section 4 presents three numerical examples to test the validity and practicality of the obtained results. The conclusion of this work is provided in Section 5.

2. Preliminaries

Definition 2.1 ([8]).
$0 < \alpha(t) < 1,\ f \in C[t_0, T]$, the variable order caputo derivtive:

$$ {}_{t_0}^{C}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_{t_0}^{t} (t-\tau)^{-\alpha(t)} f'(\tau) d\tau $$

Definition 2.2 ([8]).
the RL-integral with $\alpha(t)$ order is defined as:

$$ {}_{t_0}I_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_{t_0}^{t} (t-\tau)^{\alpha(t)-1} f(\tau) d\tau $$

where $\quad 0 < \alpha(t) < 1$ and $\Gamma(.)$ is the Gamma function as an extension of the factorial function to real numbers

$$ \Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt, \quad (Re(z) > 0) $$

where $Re(z)$ is the real part of z.

Definition 2.3 ([14]).
The constant x_0 is an equilibrium point of the variable oreder system

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha(t)} x(t) = f(t, x(t)), & t \in [t_0, T], \\ x(t_0) = x_0, \end{cases} $$

Where $0 < \gamma \leq \alpha(t) \leq \beta < 1$ if

$$ f(t, x_0) = 0 \tag{1} $$

Theorem 2.1 ([14]).
For$t_0 = 0$, the fractional-order system (6) is Mittag-Leffler stable at theequilibrium point $x^* = 0$ if there exists a continuously differentiable function $V(t, x(t))$ satisfies

$$ \begin{cases} q_1 \|x\|^a \leq V(t, x(t)) \leq q_2 \|x\|^{ab} \\ D^{\alpha(t)} V(t, x(t)) \leq -q_3 \|x\|^{ab} \end{cases} \tag{2} $$

where $V(t, x(t)) : [0, \infty) * D \to \mathbb{R}$ satisfies locally Lipschitz condition on x; $D \in \mathbb{R}^n$ is a domain containing the origin; $t \geq 0, \alpha(t) \in (0, 1)$ with $\gamma \leq \alpha(t) \leq \beta, q_1, q_2, q_3$, a and b are arbitrary positive constants. If the assumptions hold globally on $\mathbb{R}^n$, then x_0^* is globally Asymptoticly stable.

Theorem 2.2 ([14]).
Assume that $V(t)$ is a countinuous and positive definite function which satisfies

$$ {}_{t_0}^{C}D_t^{\beta} \leq -\alpha V(t) $$

for $t \geq t_0$, and $\alpha > 0$. Then, the following inequality hold

$$ V(t) \leq V(t_0) E_\beta(-\alpha t^\beta) $$

where $E_\beta(t)$ is a Mittag-Leffler function.

3. Mains results

Considering the following variable-order fractional neural networks with time varying external inputs

$$ {}_{t_0}^{C}D_t^{\alpha(t)}x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(t, x_j(t)) + I_i(t) \tag{3} $$

or equivalently

$$ {}_{t_0}^{C}D_t^{\alpha(t)}x(t) = -Cx(t) + Bg(t, x(t)) + I(t) $$

where

$0 < \alpha(t) < 1$, $C = diag(c_i)$, $c_i > 0$, $i = 1, 2, , n$, n represents the number of units in the network; $x(t) = [x_1(t), x_2(t), , x_n(t)]^T \in \mathbb{R}^n$; $B = (b_{ij})_{1\leq ij \leq n}$ corresponds to the connection of the ith neuron to the jth neuron; $g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), , g_n(x_n(t))]^T$ is the activation function of the neurons; $I(t) = [I_1(t), I_2(t), , I_n(t)]^T$ is a time-varying external bias vector.

The study of the asymptotic stability of system (3) is divided into two parts: the first is to discuss the existence and unicity of the equilibrium point using the Banach fixed point technique, and the second is to address the asymptotic stability using the Lyapunov approach.

First, we state the following requared assumptions.

Assumption (A_1) the activation function g_j, is Lipschitz continuous, i.e there exists positive constants G_j such that

$$ |g_j(x) - g_j(y)| \leq G_j |x - y|; \quad j = 1, 2, ..., n; \quad \forall x, y \in \mathbb{R} $$

Assumption (A_2) the following inequality holds

$$ -c_i + \sum_{j=1}^{n} |b_{ji}| G_i < 0, \quad i = 1, ..., n $$

3.1. Existence and uniqueness of the equilibrium point

Theorem 3.1.
if (A_1) and (A_2) are valid. Then, there exists a unique equilibrium point for system (3).

Proof. Define $\|x\| = \|x\|_1$ in the following, i.e.,

$\|x\| = \sum_{i=1}^{n} |x_i|$

for any $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$.

To beging with, constructing a mapping $\phi(u) = (\phi_1(u), \Phi_2(u), ..., \phi_n(u))^T$, where $u = (u_1, u_2, ..., u_n)$ and

$$ \phi(u) = \sum_{j=1}^{n} b_{ij} g_j(\frac{u_j}{c_j}) + I_i^*(t) $$

Considering $\forall u, v \in \mathbb{R}^n$ $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ according to (H_1), then

$$\|\phi(u) - \phi(v)\| = \sum_{i=1}^{n} |\phi_i(u) - \phi_i(v)| \leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{|b_{ij}| G_j}{c_j} |u_j - v_j| \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|b_{ji}| G_i}{c_i} |u_i - v_i|$$

According to Theorem, we obtain

$$\|\phi(u) - \phi(v)\| \leq \sum_{i=1}^{n} \left(1 - \frac{a_i}{c_i}\right) |u_i - v_i| \leq \delta \|u - v\|$$

where $\delta = \max_{i=1,...,n} \left\{1 - \frac{c_i}{a_i}\right\} < 1$ so $\|\phi(u) - \phi(v)\| < \|u - v\|$. Which implies that ϕ is a contraction mapping on $\mathbb{R}^n$. Therfor it exists a unique fixed point u^* such that $\phi(u^*) = u^*$
i.e. for $x_i^* = \frac{u_i^*}{c_i}$, then

$$-c_i x_i^* + \sum_{j=1}^{n} b_{ij} g_j(x_i^*) + I_i^*(t) = 0, \quad i = 1, 2, ..., n \tag{4}$$

Which means that x_i^* is a unique solution of (4). it is clear that x^* is the unique equilibrium point of (3).

The proof is completed. □

3.2. Stability analysis

Theorem 3.2.
if (A_1), (A_2) hold. Then, the variable-order nerual networks with time-varying external inputs (3) is globally asymptoticly stable.

Proof. Assuming that the equilibrium point of (3) is the solution of $x(t) = x^*$ we translate the equilibrium point x^* to the origin via the change of variable $e_i(t) = x_i(t) - x^*$, then we get

$${}_{t_0}^{C}D_t^{\alpha(t)} e_i(t) = -c_i e_i(t) + \sum_{j=1}^{n} b_{ij}[g_j(t, x_j(t)) - g_j(t, x_j^*)]$$

if $e_i(t) = 0$ then, ${}_{t_0}^{C}D_t^{\alpha(t)} e_i(t) = 0$

if $e_i(t) > 0$ then,

$${}_{t_0}^{C}D_t^{\alpha(t)} |e_i(t)| = \frac{1}{\Gamma(1 - \alpha(t))} \int_{t_0}^{t} (t - \tau)^{-\alpha(t)} |e_i(\tau)|' d\tau = \frac{1}{\Gamma(1 - \alpha(t))} \int_{t_0}^{t} (t - \tau)^{-\alpha(t)} e_i'(\tau) d\tau = {}_{t_0}^{C}D_t^{\alpha(t)} e_i(t)$$

if $e_i(t) < 0$ then,

$${}_{t_0}^{C}D_t^{\alpha(t)} |e_i(t)| = \frac{1}{\Gamma(1 - \alpha(t))} \int_{t_0}^{t} (t - \tau)^{-\alpha(t)} |e_i(\tau)|' d\tau = \frac{-1}{\Gamma(1 - \alpha(t))} \int_{t_0}^{t} (t - \tau)^{-\alpha(t)} e_i'(\tau) d\tau = -{}_{t_0}^{C}D_t^{\alpha(t)} e_i(t)$$

which implies

$${}_{t_0}^{C}D_t^{\alpha(t)} |e_i(t)| = sgn(e_i(t))\, {}_{t_0}^{C}D_t^{\alpha(t)} e_i(t) \tag{5}$$

Now, consider the lyapunov function

$$V(t,e(t)) = \sum_{i=1}^{n} |e_i(t)|$$

Calculate the derivative of $V(t,e(t))$ along the solution of (3) which with the use of (A_1), (A_2) and (5) lead us to the following inquality.

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha(t)}V(t,e(t)) &= \sum_{i=1}^{n} {}_{t_0}^{C}D_t^{\alpha(t)}|e_i(t)| = \sum_{i=1}^{n} sgn(e_i(t))\, {}_{t_0}^{C}D_t^{\alpha(t)}e_i(t) \\
&= \sum_{i=1}^{n} sgn(e_i(t)) \left(-c_i e_i(t) + \sum_{j=1}^{n} b_{ij}[g_j(t,x_j(t)) - g_j(t,x_j^*)] \right) \\
&\leq \sum_{i=1}^{n} -c_i|e_i(t)| + \sum_{j=1}^{n} |b_{ij}|G_j|e_j(t)| \\
&= \sum_{i=1}^{n} -c_i|e_i(t)| + \sum_{j=1}^{n} |b_{ji}|G_i|e_i(t)| \\
&\leq \delta \sum_{i=1}^{n} |e_i(t)| = \delta V(t,e(t))
\end{aligned}$$

where according to (A_2) : $\delta = \min_{i=1,\ldots,n} \left\{ -c_i + \sum_{j=1}^{n} |b_{ji}|G_i \right\} < 0$
respect to Theorem (2.1) and (2.2), we have

$$\|e(t)\| \leq [V(0)q_1^{-1}E_\beta(-\delta q_3 q_2^{-1}(t-t_0)^\beta)]^{\frac{1}{a}}$$

This indicates that $\|x(t)-x^*\|$ converges asymptotically to zero as t approaches infinity. Therefor, the equilibrium point of system (3) is asymptoticlly stable.

The proof is completed. □

3.3. Synchronization scheme

A necessary condition for synchronization of variable-order neural networks with time-varying external inputs is given in this section.

We refer to system (3) as the drive system and propose a response system described as follows:

$$ {}_{t_0}^{C}D_t^{\alpha(t)}z_i(t) = -c_i z_i(t) + \sum_{j=1}^{n} b_{ij} g_j(t,z_j(t)) + I_i(t) + v_i(t) \tag{6}$$

or equivalently

$$ {}_{t_0}^{C}D_t^{\alpha(t)}z(t) = -Cz(t) + Bg(t,z(t)) + I(t) + v(t)$$

Where $z(t) = (z_1(t), z_2(t), ..., z_n(t))^T \in \mathbb{R}^n$ is the state vector of the slave system (6) and $v(t) = (v_1(t), v_2(t), ..., v_n(t))^T$ is the external control input.

Definition 3.1 ([25]).
If any solutions $x(t)$ of (3) and $z(t)$ of (6) satisfy the condition

$$\lim_{t \to +\infty} \|x(t) - y(t)\| = 0$$

then systems (3) and (6) are said to be global asymptotic synchronization.

Now, identifying the synchronization error as $e_i(t) = z_i(t) - x_i(t)$. The error characteristics between the master network (3) and slave network (6) may be represented as:

$$ {}_{t_0}^{C}D_t^{\alpha(t)} e(t) = -Ce(t) + B(g(t, z(t)) - g(t, x(t))) + v(t) \tag{7}$$

Synchronization between master system (3) and slave system (6) is comparable to the asymptotic stability of error system (7) with the appropriate control law $v(t)$. In this context, the external control input $v(t)$ can be set to $v(t) = Ae(t)$, where $A = diag(a_1, a_2, ..., a_n)$ is the controller gain matrix.
Error system (7) becomes

$$ {}_{t_0}^{C}D_t^{\alpha(t)} e(t) = -(C - A)e(t) + B(g(t, z(t)) - g(t, x(t))) \tag{8}$$

Theorem 3.3.
Assume that (A_1) is fulfiled and the system parameters satisfy

$$a_i < c_i - \sum_{j=1}^{n} |b_{ji}| G_i, \quad i = 1, ..., n$$

Then, the drive system (3) and the corresponding response system (6) are globally assymptoticly synchronized.

Proof. Consider the function $V(t, e(t)) = \sum_{i=1}^{n} |e_i(t)|$
The Caputo variable-order derivatirve along the solution of the error system (8) is

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha(t)} V(t, e(t)) &= \sum_{i=1}^{n} {}_{t_0}^{C}D_t^{\alpha(t)} |e_i(t)| = \sum_{i=1}^{n} sgn(e_i(t))\, {}_{t_0}^{C}D_t^{\alpha(t)} e_i(t) \\
&= \sum_{i=1}^{n} sgn(e_i(t)) \left(-(c_i - a_i)e_i(t) + \sum_{j=1}^{n} b_{ij}[g_j(t, z_j(t)) - g_j(t, x_j)] \right) \\
&\leq \sum_{i=1}^{n} -(c_i - a_i)|e_i(t)| + \sum_{j=1}^{n} |b_{ij}| G_j |e_j(t)| \\
&= \sum_{i=1}^{n} \left\{ -(c_i - a_i) + \sum_{j=1}^{n} |b_{ji}| G_i \right\} |e_i(t)| \\
&\leq \lambda \sum_{i=1}^{n} |e_i(t)| = \lambda V(t, e(t))
\end{aligned}$$

where $\lambda = \min_{i=1,\dots,n}\{-(c_i - a_i) + \sum_{j=1}^{n} |b_{ji}|G_i\} < 0$
respect to Theorem (2.1) and (2.2), we conclude that

$$\|e(t)\| \leq [V(t_0)q_1^{-1}E_\beta(-\lambda q_3 q_2^{-1}(t-t_0)^\beta)]^{\frac{1}{a}}$$

Therefor, $\|z(t) - x(t)\|$ converges asymptotically to zero as $\to +\infty$, implying that the variable-order neural networks (3) is globally synchronized with system (6). □

4. Numerical examples

An exemple is presented in this section so as to illustrate the validity of the theoriticals results concerning the stablity and synchronization of the variable fractional-order neural networks. The numerical solution of the considered variable-order system is calculated by using the Adams-Bashforth-Moulton method [23].

Example 1 Consider the fractional variable-order system with $n = 3$ and the variable order function $\alpha(t) = e^{(t-1)} - \frac{t}{5}$;

where $t \in [-1.5, 1]$; $x(t) = (x_1(t), x_2(t), x_3(t))^T$; $g(t, x(t)) = \left(\sin\left(\frac{x(t)}{2}\right), \sin(x(t)), \sin\left(\frac{x(t)}{2}\right)\right)^T$;

$C = diag(2, 2, 2)$; $I(t) = \left(-\frac{t}{6}, -\frac{e^{-t^3}}{2}, -\sqrt{\frac{|t^3|}{3}}\right)^T$ and $B = \begin{bmatrix} -0.4 & 0.1 & -0.9 \\ 0.1 & -0.3 & -0.8 \\ -0.4 & 0.6 & -0.2 \end{bmatrix}$

the variable-order neural networks is described as

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha(t)} x_1(t) = -2x_1(t) - 0.4\sin\left(\frac{x_1(t)}{2}\right) + 0.1\sin\left(\frac{x_2(t)}{2}\right) - 0.9\sin\left(\frac{x_3(t)}{2}\right) - \frac{t}{6} \\ {}_{t_0}^{C}D_t^{\alpha(t)} x_2(t) = -2x_1(t) + 0.1\sin(x_1(t)) - 0.3\sin(x_2(t)) - 0.8\sin(x_3(t)) - \frac{e^{-t^3}}{2} \\ {}_{t_0}^{C}D_t^{\alpha(t)} x_2(t) = -2x_1(t) - 0.4\sin\left(\frac{x_1(t)}{2}\right) + 0.6\sin\left(\frac{x_2(t)}{2}\right) - 0.2\sin\left(\frac{x_3(t)}{2}\right) - \sqrt{\frac{|t^3|}{3}} \end{cases} \tag{9}$$

With the initial condition $x_1(-1.5) = 1,\ x_2(-1.5) = 1,\ x_3(-1.5) = 1$

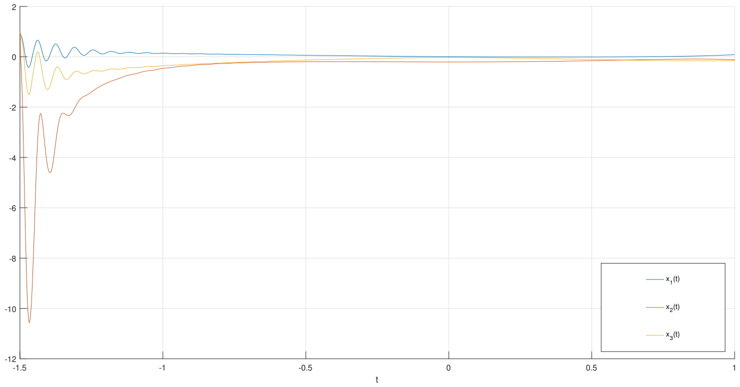
the conditions of the two Theorems discussed above hold obviously with respect to the parameters of system Therefor the asymtptic stability is ensured.

The numerical simulation Figure (1) shows the analyzied stability of (9)

Exemple 2 Consider the variable fractional-order neural networks

$$\begin{cases} {}_{0}^{C}D_t^{\alpha(t)} x_1(t) = -c_1x_1(t) + b_{11}\sin(x_1(t)) + b_{12}\sin(x_2(t)) + I_1 \\ {}_{0}^{C}D_t^{\alpha(t)} x_2(t) = -c_2x_1(t) + b_{21}\sin(x_1(t)) + b_{22}\sin(x_2(t)) + I_2 \end{cases} \tag{10}$$

Figure 1. **Numerical solution of variable-order neural networks with time depending external inputs (9)**



where the external inputs are taken as $I_1 = -1,\ I_2 = -2$ and the parameters of the system are $c_1 = 7,\ c_2 = 6.5$ and

$$B = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

and the variable order function

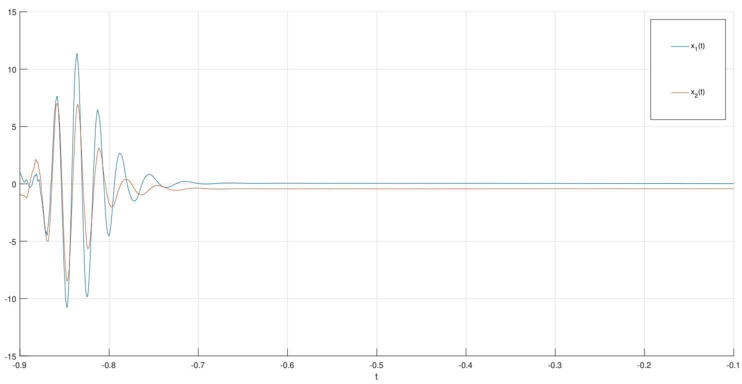
$$\alpha(t) = \sqrt{(t+1)}, \quad t \in [-0.9, -0.1]$$

The activation functions are chossen as $g_1(x) = g_2(x) = \sin(x)$. With the initial condition $x_1(-0.9) = 1,\ x_2(-0.9) = -1$

the assumptions (A_1) and (A_2) hold obviously with respect to the parameters of system Therefor the stability is ensured.

Figure (2) shows that the solution of system (10) converges to the equilibrium point.

Figure 2. **Numerical solution of variable-order neural networks (10)**



Example 3 Let be the following variable-order chaotic neural network with time varying external inputs

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha(t)}x_1(t) = -c_1x_1(t) + b_{11}\tanh(x_1(t)) + b_{12}\tanh(x_2(t)) + I_1(t) \\ {}_{t_0}^{C}D_t^{\alpha(t)}x_2(t) = -c_2x_1(t) + b_{21}\tanh(x_1(t)) + b_{22}\tanh(x_2(t)) + I_2(t) \end{cases} \tag{11}$$

where

$$C = diag(3,4), \quad B = \begin{bmatrix} -2 & 1.5 \\ -3 & 1.5 \end{bmatrix}, \quad I(t) = \left(\frac{-\sin(t)}{5}, -\cos(t)\right)^T$$

As seen in Figure (3), the variable order neural networks (11) exhibit a chaotic behavior when

$$\alpha(t) = 1 - \frac{e^{-2t}}{4}$$

The controlled response system is as follow

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha(t)}z_1(t) = -c_1z_1(t) + b_{11}\tanh(z_1(t)) + b_{12}\tanh(z_2(t)) + I_1(t) + v_1(t) \\ {}_{t_0}^{C}D_t^{\alpha(t)}z_2(t) = -c_2z_1(t) + b_{21}\tanh(z_1(t)) + b_{22}\tanh(z_2(t)) + I_2(t) + v_2(t) \end{cases} \tag{12}$$

The controller gain matrix is $K = diag(-1, -1.5)$. According to Theorem the sunchronization between (11) and (12) can be achived.

the initial conditions of the master and slave systems are given by $x(0.5) = (1,1)^T,\ y(0.5) = (0.1, 0.1)^T$

The numerical simulations shows the state synchronization trajectory of the master-slave systems in Figure (4) and in Figure (5) we have the synchronization errors.

Figure 3. **Chaotic behavior variable order neural networks (11**

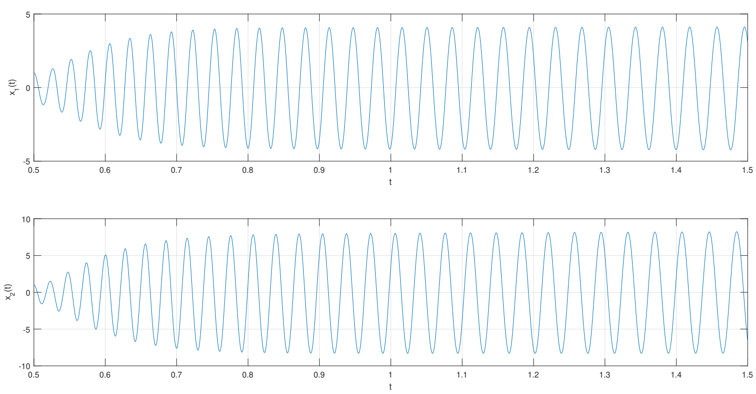


Figure 4. **the state synchronization trajectory of the master-slave systems**

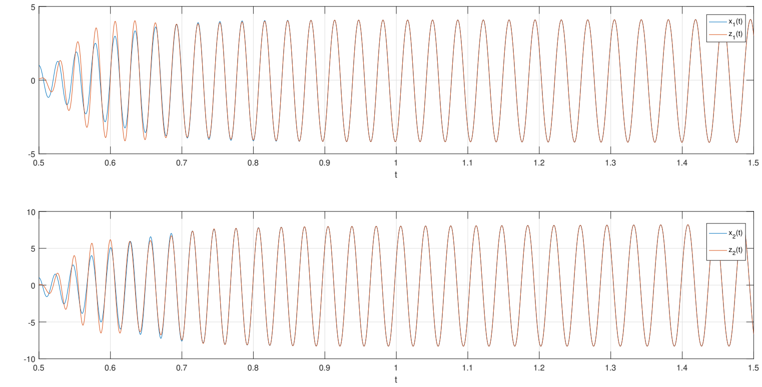
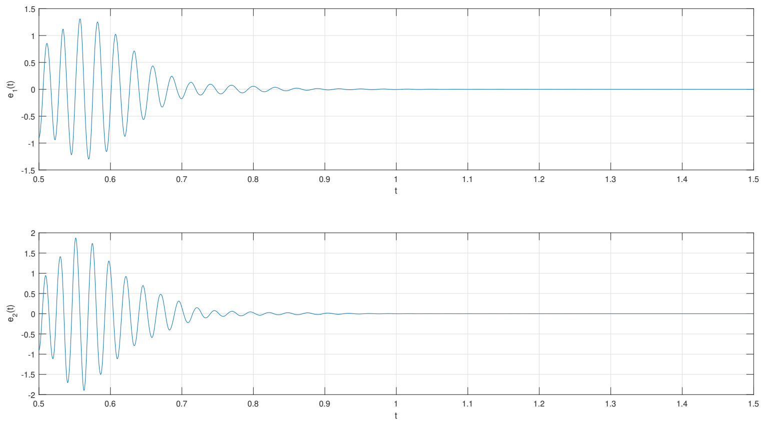


Figure 5. **synchronization error**



5. Conclusion

This study is devoted to investigating the existence, uniqueness, asymptotic stability and synchronization of variable-order fractioanl neural networks with time-varying external inputs with the use of the Caputo variable-order fractional derivative. Based on variable fractional calculus and the extension of the Lyapunov direct method to variable order neural networks case, necessary conditions are provided to establishe the asymptotic stability and global asymptotic synchronization of the considered neural networks. Three numerical examples are provided to demonstrate the efficacity of our outcomes. It should be noted that our findings are easily applicable to obtaining the theoritical analysis requirements.

It has been noticed that the process and findings obtained in this paper not only provide a realistic means of understanding the behavior of a system that include variable fractional orders, but even offers the possibility of pursuing a theory of more complex and difficult variable fractional order systems in a similar direction.

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